Just as for line integrals, there is a rule of thumb which tells you when to stop using what you know to compute surface integrals: Don't start integrating until the integral is expressed in terms of *two* parameters, and the limits in terms of those parameters have been determined. Surfaces are two-dimensional!

5 Highly symmetric surfaces

One of the most fundamental examples in electromagnetism is the electric field of a point charge q at the origin, which is given by

$$\vec{\boldsymbol{E}} = \frac{q}{4\pi\epsilon_0}\frac{\hat{\boldsymbol{r}}}{r^2} = \frac{q}{4\pi\epsilon_0}\frac{x\,\hat{\boldsymbol{\imath}} + y\,\hat{\boldsymbol{\jmath}} + z\,\hat{\boldsymbol{k}}}{(x^2+y^2+z^2)^{3/2}} \tag{16}$$

where $\hat{\boldsymbol{r}}$ is now the unit vector in the radial direction in *spherical* coordinates. Note that the first expression clearly indicates both the spherical symmetry of $\vec{\boldsymbol{E}}$ and its $\frac{1}{r^2}$ fall-off behavior, while the second expression does neither. Given the electric field, Gauss' Law allows one to determine the total charge inside any closed surface, namely

$$\frac{q}{\epsilon_0} = \iint\limits_S \vec{\boldsymbol{E}} \cdot d\vec{\boldsymbol{S}} \tag{17}$$

which is of course just the Divergence Theorem.

As noted in the introduction, it is easy to determine $d\vec{\boldsymbol{S}}$ on the sphere by inspection; we nevertheless go through the details of the differential approach for this case. We use "physicists' conventions" for spherical coordinates, so that θ is the angle from the North Pole, and ϕ the angle in the xy-plane; see [6]. We use the obvious families of curves, namely the lines of latitude and longitude. Starting either from the general formula for $d\vec{\boldsymbol{r}}$ in spherical coordinates, namely

$$d\vec{\boldsymbol{r}} = dr\,\hat{\boldsymbol{r}} + r\,d\theta\,\hat{\boldsymbol{\theta}} + r\sin\theta\,d\phi\,\hat{\boldsymbol{\phi}} \tag{18}$$

or directly using the geometry behind that formula, one quickly arrives at

$$d\vec{\boldsymbol{S}} = d\vec{\boldsymbol{r}}_1 \times d\vec{\boldsymbol{r}}_2 = r\,d\theta\,\hat{\boldsymbol{\theta}} \times r\sin\theta\,d\phi\,\hat{\boldsymbol{\phi}} \tag{19}$$

so that

$$\iint\limits_S \vec{\boldsymbol{E}} \cdot d\vec{\boldsymbol{S}} = \int_0^{2\pi}\int_0^{\pi} \frac{q}{4\pi\epsilon_0}\frac{\hat{\boldsymbol{r}}}{r^2} \cdot r^2\sin\theta\,d\theta\,d\phi\,\hat{\boldsymbol{r}} = \frac{q}{\epsilon_0} \tag{20}$$