BRIDGING THE GAP between Mathematics and the Physical Sciences

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I: Mathematics $\neq$ Physics
II: The Bridge Project

http://www.math.oregonstate.edu/bridge

TODAY

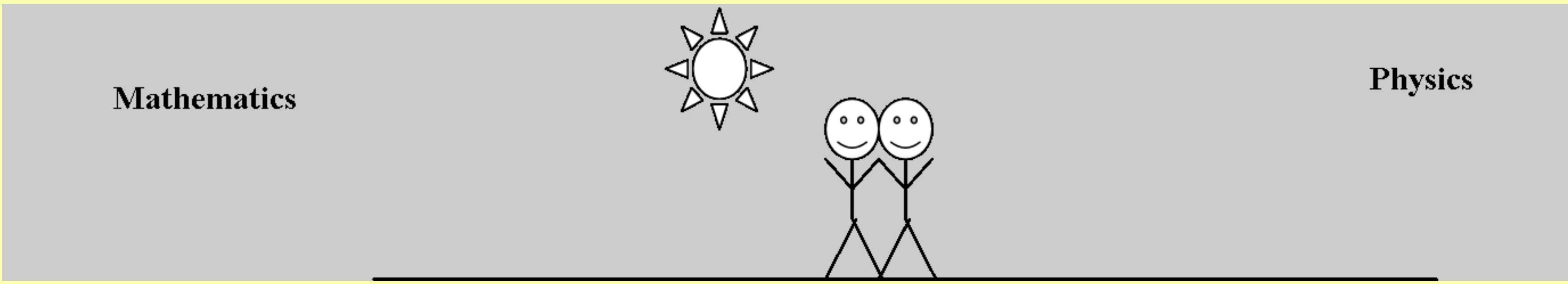
What is special about today, 14 March 2009?

Choices

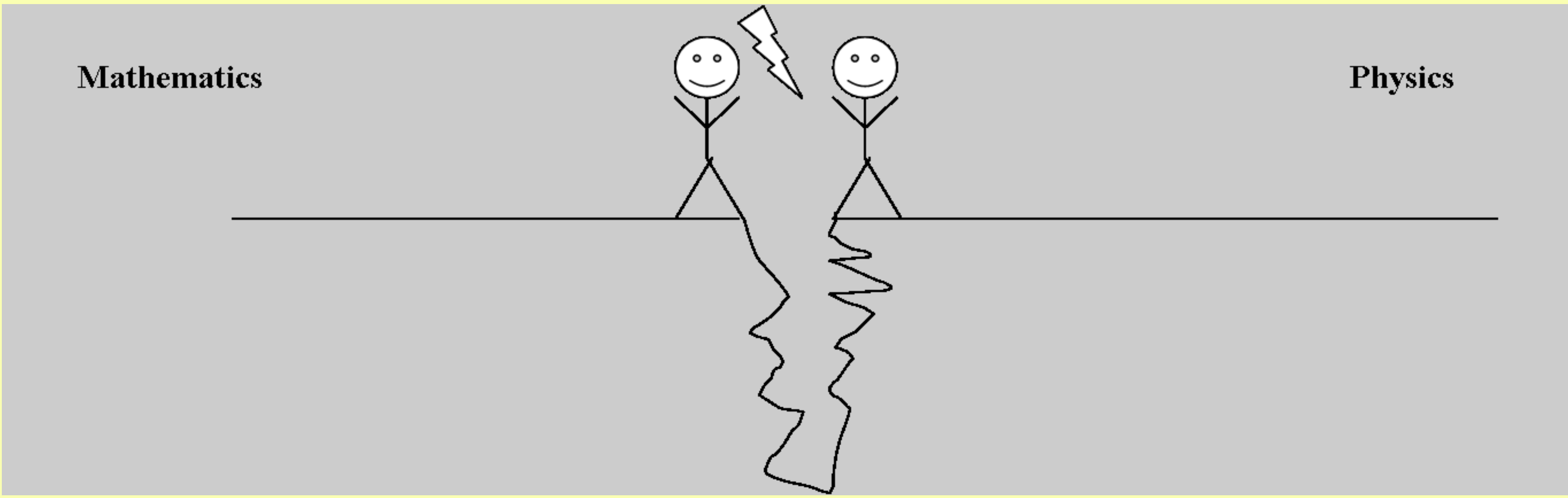
A: π day

B: Einstein's birthday

Mathematics vs. Physics



Mathematics vs. Physics



FUNCTIONS

Suppose the temperature on a rectangular slab of metal is given by $T(x, y) = k(x^2+y^2)$ where k is a constant. What is $T(r, \theta)$?

Choices

A: $T(r, \theta) = kr^2$

B: $T(r, \theta) = k(r^2 + \theta^2)$

MATH

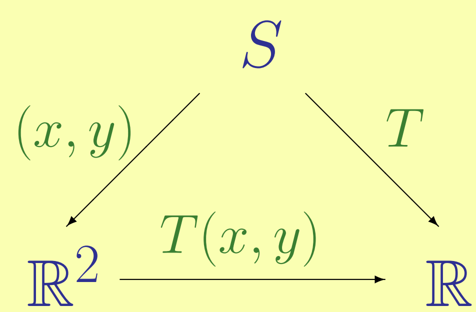
$$T = f(x,y) = k(x^2+y^2)$$
$$T = g(r,\theta) = kr^2$$

PHYSICS

$$T = T(x,y) = k(x^2+y^2)$$
$$T = T(r,\theta) = kr^2$$

Differential Geometry!

$$T(x,y) \longleftrightarrow T\circ(x,y)^{-1}$$
$$T(r,\theta) \longleftrightarrow T\circ(r,\theta)^{-1}$$



DOT PRODUCT

MATHEMATICIANS' LINE INTEGRALS

- Start with Theory

$$\begin{aligned}\int \vec{\boldsymbol{F}} \cdot d\vec{\boldsymbol{r}} &= \int \vec{\boldsymbol{F}} \cdot \hat{\boldsymbol{T}}\, ds \\ &= \int \vec{\boldsymbol{F}}(\vec{\boldsymbol{r}}(t)) \cdot \frac{\vec{\boldsymbol{r}}\,'(t)}{|\vec{\boldsymbol{r}}\,'(t)|} |\vec{\boldsymbol{r}}\,'(t)|\, dt \\ &= \int \vec{\boldsymbol{F}}(\vec{\boldsymbol{r}}(t)) \cdot \vec{\boldsymbol{r}}\,'(t)\, dt \\ &= \ldots = \int P\, dx + Q\, dy + R\, dz\end{aligned}$$

- Do examples starting from next-to-last line

Need parameterization $\vec{\boldsymbol{r}} = \vec{\boldsymbol{r}}(t)$

MATHEMATICS

$$\vec{\boldsymbol{F}}(x,y) = \frac{-y\,\hat{\boldsymbol{\imath}} + x\,\hat{\boldsymbol{\jmath}}}{x^2+y^2} \qquad \vec{\boldsymbol{r}} = x\,\hat{\boldsymbol{\imath}} + y\,\hat{\boldsymbol{\jmath}}$$

$$\begin{aligned} x &= 2\cos\theta \\ y &= 2\sin\theta \end{aligned}$$

$$\begin{aligned} \int \vec{\boldsymbol{F}}\cdot d\vec{\boldsymbol{r}} &= \int_0^{\frac{\pi}{2}} \vec{\boldsymbol{F}}(x(\theta),y(\theta))\cdot \vec{\boldsymbol{r}}\,'(x(\theta),y(\theta))\,d\theta \\ &= \int_0^{\frac{\pi}{2}} \frac{1}{2}(-\sin\theta\,\hat{\boldsymbol{\imath}} + \cos\theta\,\hat{\boldsymbol{\jmath}})\cdot 2(-\sin\theta\,\hat{\boldsymbol{\imath}} + \cos\theta\,\hat{\boldsymbol{\jmath}})\,d\theta \\ &= \ldots = \frac{\pi}{2} \end{aligned}$$

COORDINATES

What is $\vec{\boldsymbol{r}}$?

Choices

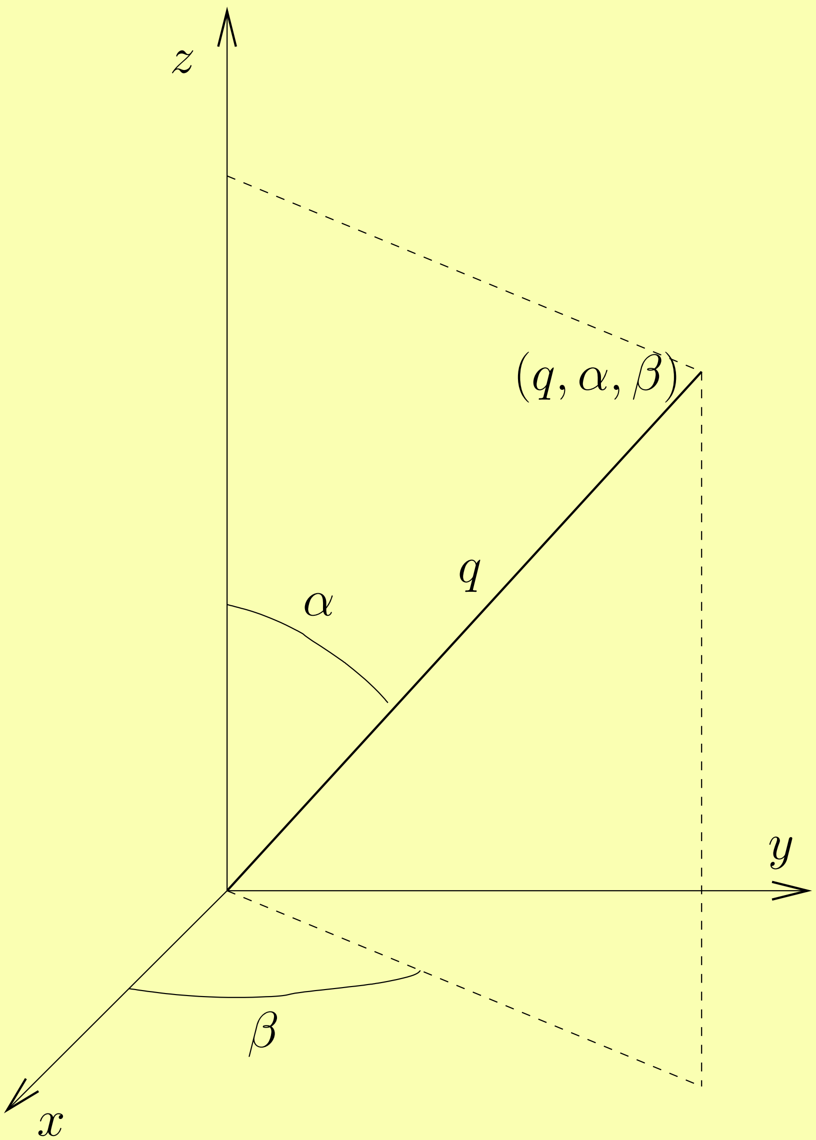
A: $\vec{\boldsymbol{r}} = x\,\hat{\boldsymbol{\imath}} + y\,\hat{\boldsymbol{\jmath}}$

B: $\vec{\boldsymbol{r}} = r\cos\theta\,\hat{\boldsymbol{\imath}} + r\sin\theta\,\hat{\boldsymbol{\jmath}}$

C: $\vec{\boldsymbol{r}} = r\,\hat{\boldsymbol{r}} + \theta\,\hat{\boldsymbol{\theta}}$

D: $\vec{\boldsymbol{r}} = r\,\hat{\boldsymbol{r}}$

SPHERICAL COORDINATES



q: Radius
α: Zenith (colatitude)
β: Azimuth (longitude)

Math: (ρ, ϕ, θ)
Physics: (r, θ, ϕ)

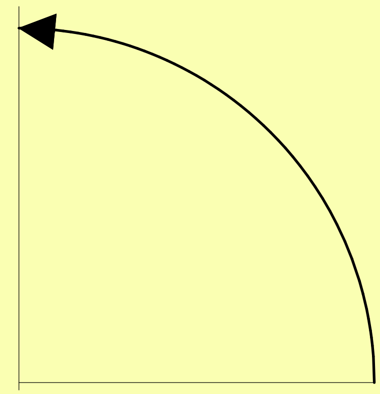
PHYSICISTS' LINE INTEGRALS

- Theory
  - Chop up curve into little pieces $d\vec{\boldsymbol{r}}$.
  - Add up components of $\vec{\boldsymbol{F}}$ parallel to curve (times length of $d\vec{\boldsymbol{r}}$)
- Do examples directly from $\vec{\boldsymbol{F}} \cdot d\vec{\boldsymbol{r}}$

Need $d\vec{\boldsymbol{r}}$ along curve

PHYSICS

$$\vec{\boldsymbol{F}} = \frac{\hat{\boldsymbol{\phi}}}{r}$$



$$d\vec{\boldsymbol{r}} = r\,d\phi\,\hat{\boldsymbol{\phi}}$$

I: $|\vec{\boldsymbol{F}}| = \text{const}; \ \vec{\boldsymbol{F}} \parallel d\vec{\boldsymbol{r}} \implies$

$$\int \vec{\boldsymbol{F}} \cdot d\vec{\boldsymbol{r}} = \frac{1}{2}(2\ \frac{\pi}{2})$$

II: Do the dot product $\longmapsto$

$$\int \vec{\boldsymbol{F}} \cdot d\vec{\boldsymbol{r}} = \int_0^{\frac{\pi}{2}} \frac{\hat{\boldsymbol{\phi}}}{2} \cdot 2\,d\phi\,\hat{\boldsymbol{\phi}} = \int_0^{\frac{\pi}{2}} d\phi = \frac{\pi}{2}$$

THE BRIDGE PROJECT

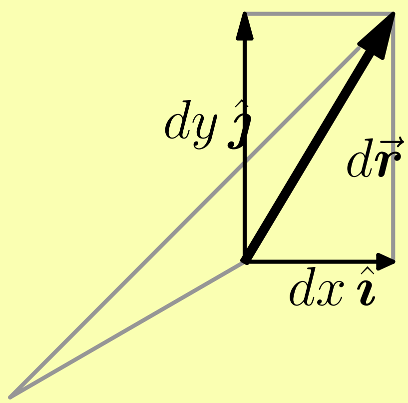
- Small group activities
- Instructor's guide
- Study Guide
- CWU, MHC, OSU, UPS, UWEC
- Workshops
- `http://www.math.oregonstate.edu/bridge`

- **Differentials** *(Use what you know!)*
- **Multiple representations**
- **Symmetry** *(adapted bases, coordinates)*
- **Geometry** *(vectors, div, grad, curl)*

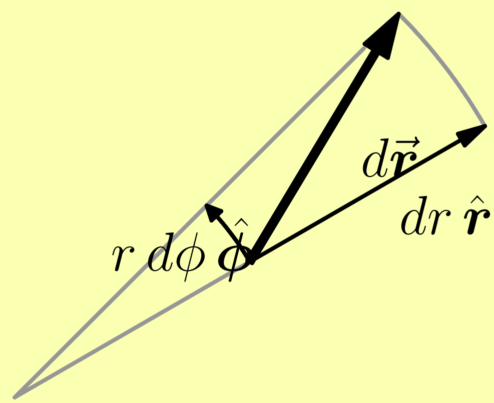
PUBLICATIONS

1. Tevian Dray and Corinne A. Manogue, *The Vector Calculus Gap: Mathematics ≠ Physics*, PRIMUS **9**, 21–28 (1999).
2. Tevian Dray and Corinne A. Manogue, *Electromagnetic Conic Sections*, Am. J. Phys. **70**, 1129–1135 (2002).
3. Tevian Dray & Corinne A. Manogue, *Spherical Coordinates*, College Math. J. **34**, 168–169 (2003).
4. Tevian Dray & Corinne A. Manogue, *The Murder Mystery Method for Determining Whether a Vector Field is Conservative*, College Math. J. **34**, 238–241 (2003).
5. Tevian Dray and Corinne A. Manogue, *Using Differentials to Bridge the Vector Calculus Gap*, College Math. J. **34**, 283–290 (2003).

VECTOR DIFFERENTIALS

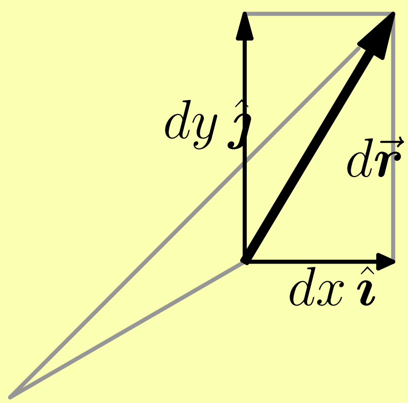


$$d\vec{\boldsymbol{r}} = dx\,\hat{\boldsymbol{\imath}} + dy\,\hat{\boldsymbol{\jmath}}$$

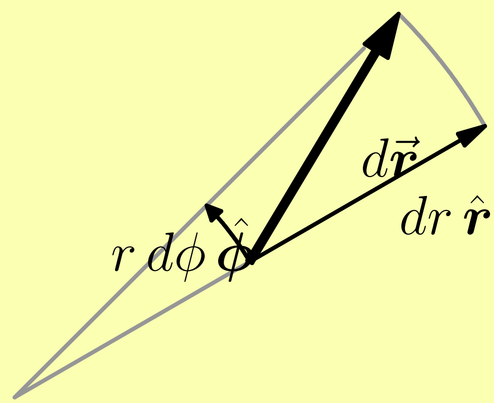


$$d\vec{\boldsymbol{r}} = dr\,\hat{\boldsymbol{r}} + r\,d\phi\,\hat{\boldsymbol{\phi}}$$

VECTOR DIFFERENTIALS



$$d\vec{\boldsymbol{r}} = dx\,\hat{\boldsymbol{\imath}} + dy\,\hat{\boldsymbol{j}}$$



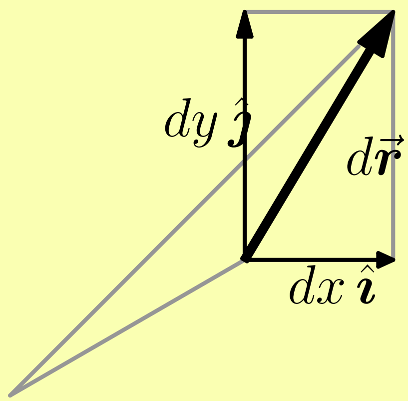
$$d\vec{\boldsymbol{r}} = dr\,\hat{\boldsymbol{r}} + r\,d\phi\,\hat{\boldsymbol{\phi}}$$

$$d\vec{\boldsymbol{r}} = dx\,\hat{\boldsymbol{\imath}} + dy\,\hat{\boldsymbol{j}} + dz\,\hat{\boldsymbol{k}}$$

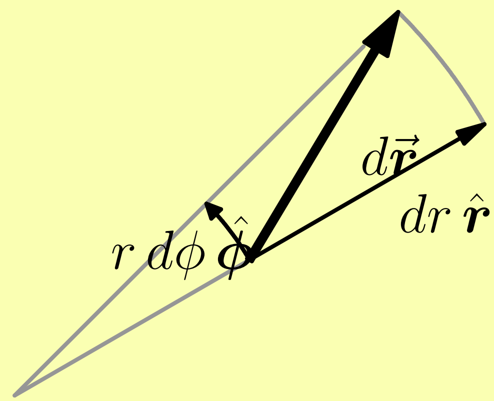
$$d\vec{\boldsymbol{r}} = dr\,\hat{\boldsymbol{r}} + r\,d\phi\,\hat{\boldsymbol{\phi}} + dz\,\hat{\boldsymbol{z}}$$

$$d\vec{\boldsymbol{r}} = dr\,\hat{\boldsymbol{r}} + r\,d\theta\,\hat{\boldsymbol{\theta}} + r\,\sin\theta\,d\phi\,\hat{\boldsymbol{\phi}}$$

VECTOR DIFFERENTIALS



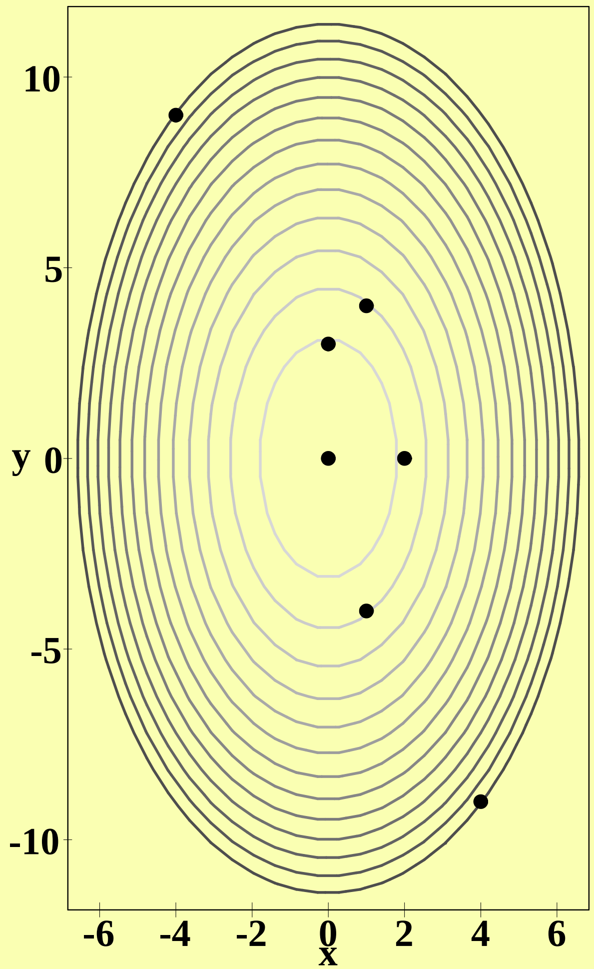
$$d\vec{\boldsymbol{r}} = dx\,\hat{\boldsymbol{\imath}} + dy\,\hat{\boldsymbol{\jmath}}$$



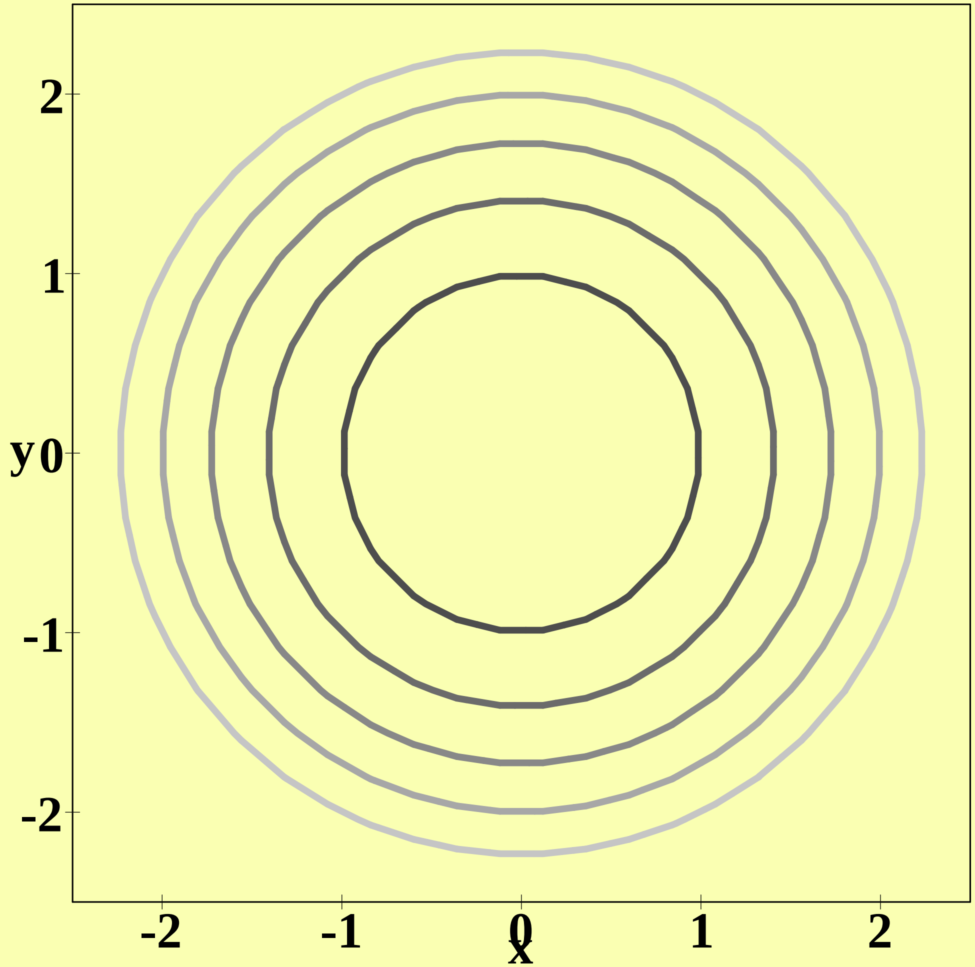
$$d\vec{\boldsymbol{r}} = dr\,\hat{\boldsymbol{r}} + r\,d\phi\,\hat{\boldsymbol{\phi}}$$

$$\begin{aligned}
ds &= |d\vec{\boldsymbol{r}}| \\
d\vec{\boldsymbol{A}} &= d\vec{\boldsymbol{r}}_1 \times d\vec{\boldsymbol{r}}_2 \\
dS &= |d\vec{\boldsymbol{r}}_1 \times d\vec{\boldsymbol{r}}_2| \\
dV &= (d\vec{\boldsymbol{r}}_1 \times d\vec{\boldsymbol{r}}_2) \cdot d\vec{\boldsymbol{r}}_3 \\
df &= \vec{\nabla} f \cdot d\vec{\boldsymbol{r}}
\end{aligned}$$

THE HILL



THE VALLEY



Mathematics $\neq$ Physics

I: Physics is about things.

II: Physicists can't change the problem.

I: Physics is about things.

(a) What sort of a beast is it?

- Is it a scalar or a vector?
- What are the units? (x is a length!)

(b) Physics is independent of coordinates.

- Vectors are arrows in space, not triples of numbers.
- Dot and cross products defined (and computed) geometrically.

(c) Graphs are about the relationships of physical things.

- Most physics is three-dimensional.
- Points in the domain of a function physically exist.
- The values of a function represent some measurable quantity.
- Hills are not good examples of functions of two variables.
- 3-d graphs of functions of 2 variables are misleading. (Use color!)

(d) Fundamental physics is highly symmetric.

- Spheres and cylinders vs. paraboloids.
- Interesting physics problems can involve trivial math.
- Use of $\hat{\boldsymbol{r}}$, $\hat{\boldsymbol{\theta}}$, $\hat{\boldsymbol{\phi}}$.

II: Physicists can't change the problem.

(a) Physics involves the creative synthesis of multiple ideas.

- Upper division vs. lower division.

(b) Physics problems may not be well-defined math problems.

- There may be no obvious coordinates or independent variables.
- A physical object rarely comes with a parameterization.
- The unknowns don't have names.
- Getting to a well-defined math problem is part of the problem.
- If you can't add units, it wouldn't make a good physics problem.

(c) Physics problems don't fit templates.

- Template problems not very useful.
- A few key ideas are more easily remembered later.

(d) Physics involves the interplay of multiple representations.

- Dot product.

THE GEOMETRY OF GRADIENT

$$
\begin{aligned}
df &= \frac{\partial f}{\partial x}\, dx + \frac{\partial f}{\partial y}\, dy + \frac{\partial f}{\partial z}\, dz \\
&= \left(\frac{\partial f}{\partial x}\, \hat{\boldsymbol{\imath}} + \frac{\partial f}{\partial y}\, \hat{\boldsymbol{\jmath}} + \frac{\partial f}{\partial z}\, \hat{\boldsymbol{k}} \right) \cdot \left(dx\, \hat{\boldsymbol{\imath}} + dy\, \hat{\boldsymbol{\jmath}} + dz\, \hat{\boldsymbol{k}} \right)
\end{aligned}
$$

$$
\boxed{df = \vec{\nabla} f \cdot d\vec{\boldsymbol{r}}}
$$

THE MASTER FORMULA

$$df = \vec{\nabla} f \cdot d\vec{\boldsymbol{r}}$$

$$f = \text{constant} \Longrightarrow df = 0 \Longrightarrow \vec{\nabla} f \perp \text{level curve}$$

$$\frac{df}{ds} = \vec{\nabla} f \cdot \frac{d\vec{\boldsymbol{r}}}{ds} = \vec{\nabla} f \cdot \frac{d\vec{\boldsymbol{r}}}{|d\vec{\boldsymbol{r}}|} = \vec{\nabla} f \cdot \hat{\boldsymbol{T}} = D_{\hat{\boldsymbol{T}}} f$$

$$\frac{df}{dt} = \vec{\nabla} f \cdot \frac{d\vec{\boldsymbol{r}}}{dt} = \vec{\nabla} f \cdot \vec{\boldsymbol{v}}$$

Polar Coordinates:

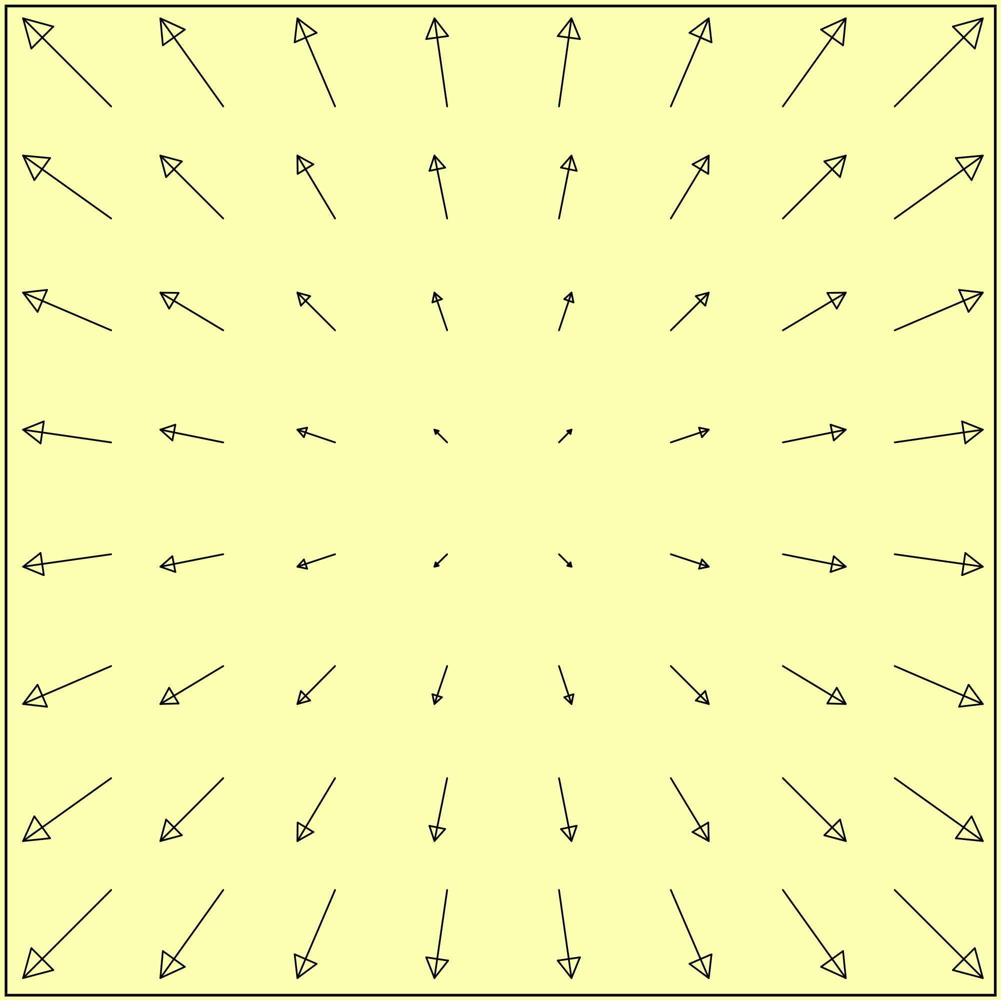
$$\begin{aligned} df &= \frac{\partial f}{\partial r} dr + \frac{\partial f}{\partial \phi} d\phi \\ d\vec{\boldsymbol{r}} &= dr\,\hat{\boldsymbol{r}} + r\,d\phi\,\hat{\boldsymbol{\phi}} \end{aligned} \qquad \Longrightarrow \qquad \vec{\nabla} f = \frac{\partial f}{\partial r}\,\hat{\boldsymbol{r}} + \frac{1}{r}\frac{\partial f}{\partial \phi}\,\hat{\boldsymbol{\phi}}$$

THE GEOMETRY OF DIVERGENCE

$$\int_{\text{box}} \vec{\boldsymbol{F}} \cdot d\vec{\boldsymbol{A}} = \int_{\text{inside}} \vec{\nabla} \cdot \vec{\boldsymbol{F}}\, dV$$

$$\vec{\nabla} \cdot \vec{\boldsymbol{F}} \approx \frac{\int \vec{\boldsymbol{F}} \cdot d\vec{\boldsymbol{A}}}{\text{volume of box}} = \frac{\text{flux}}{\text{unit volume}}$$

THE GEOMETRY OF DIVERGENCE



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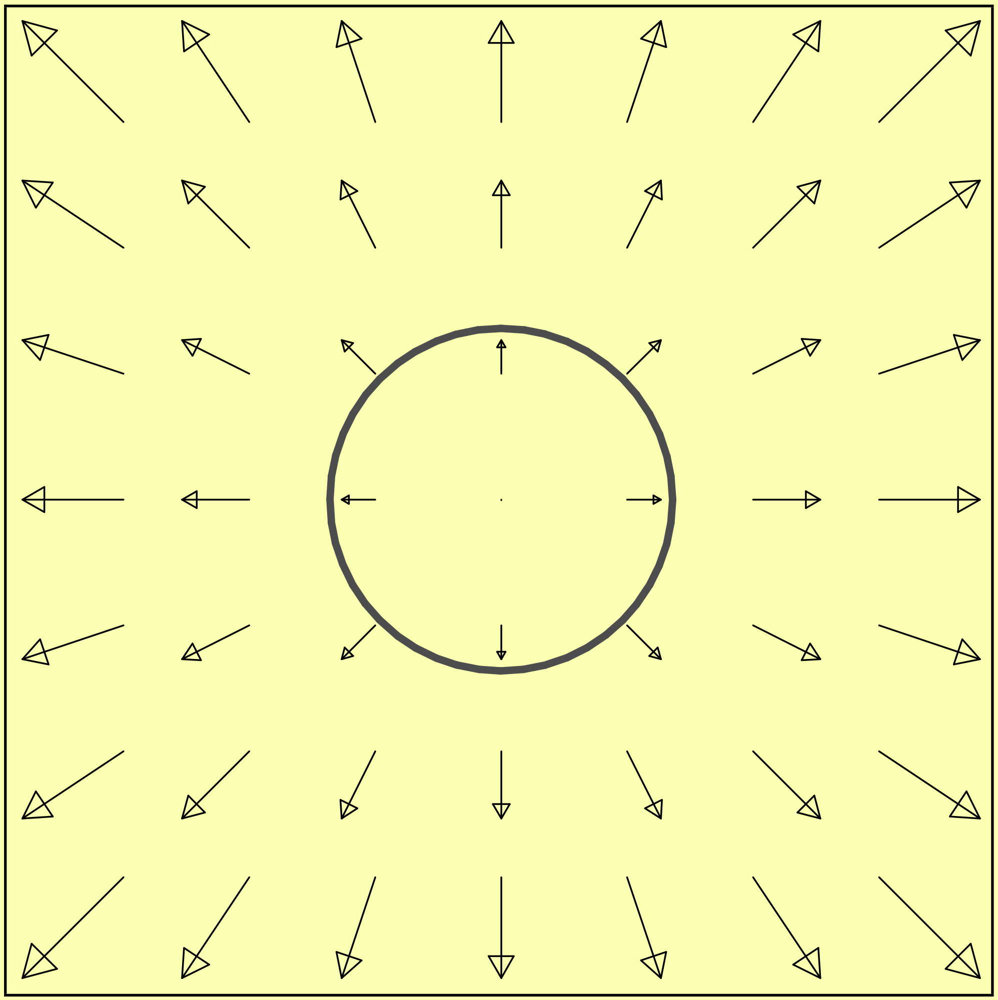

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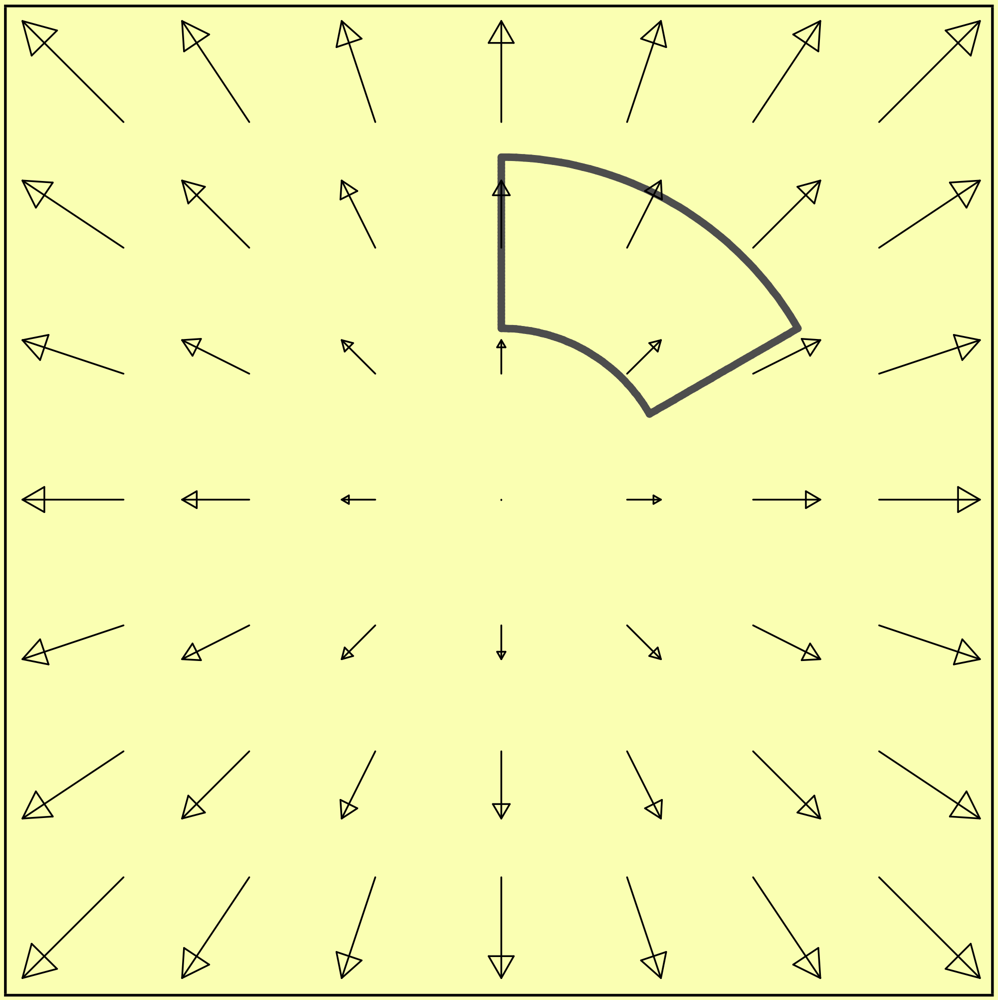

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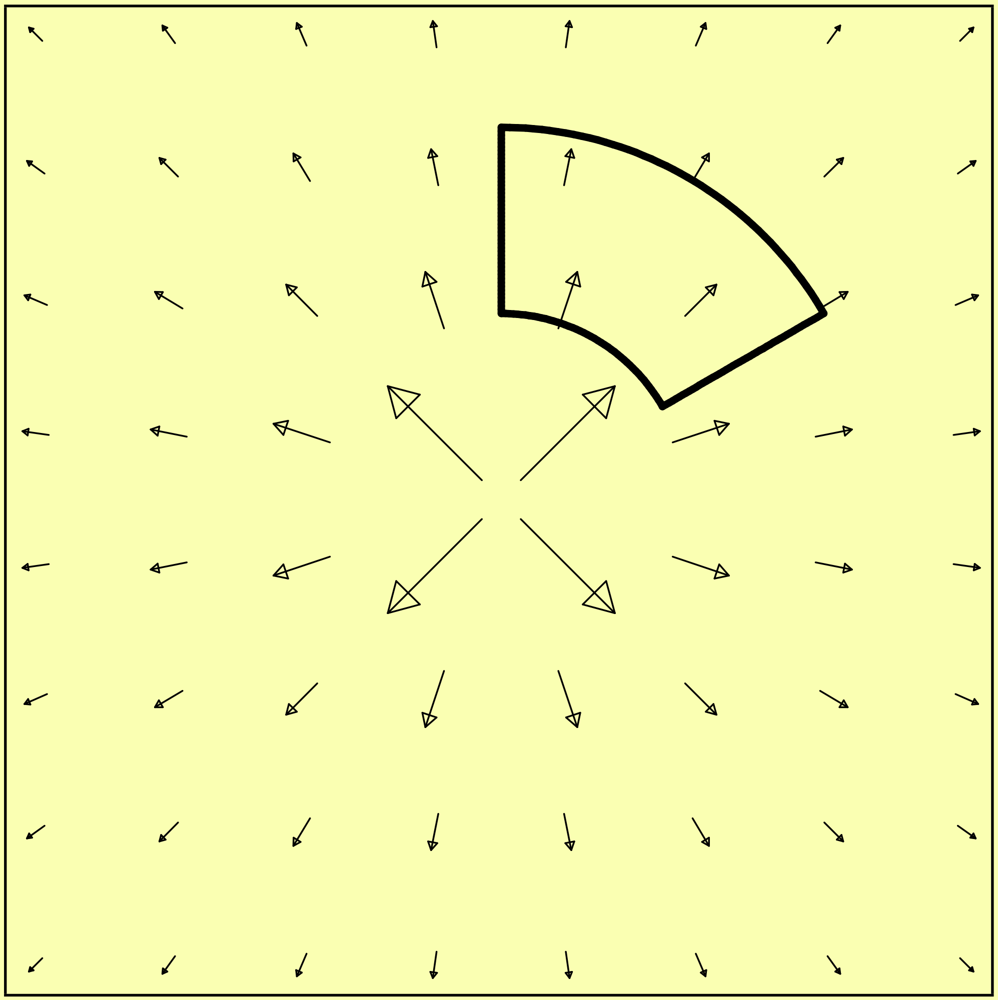
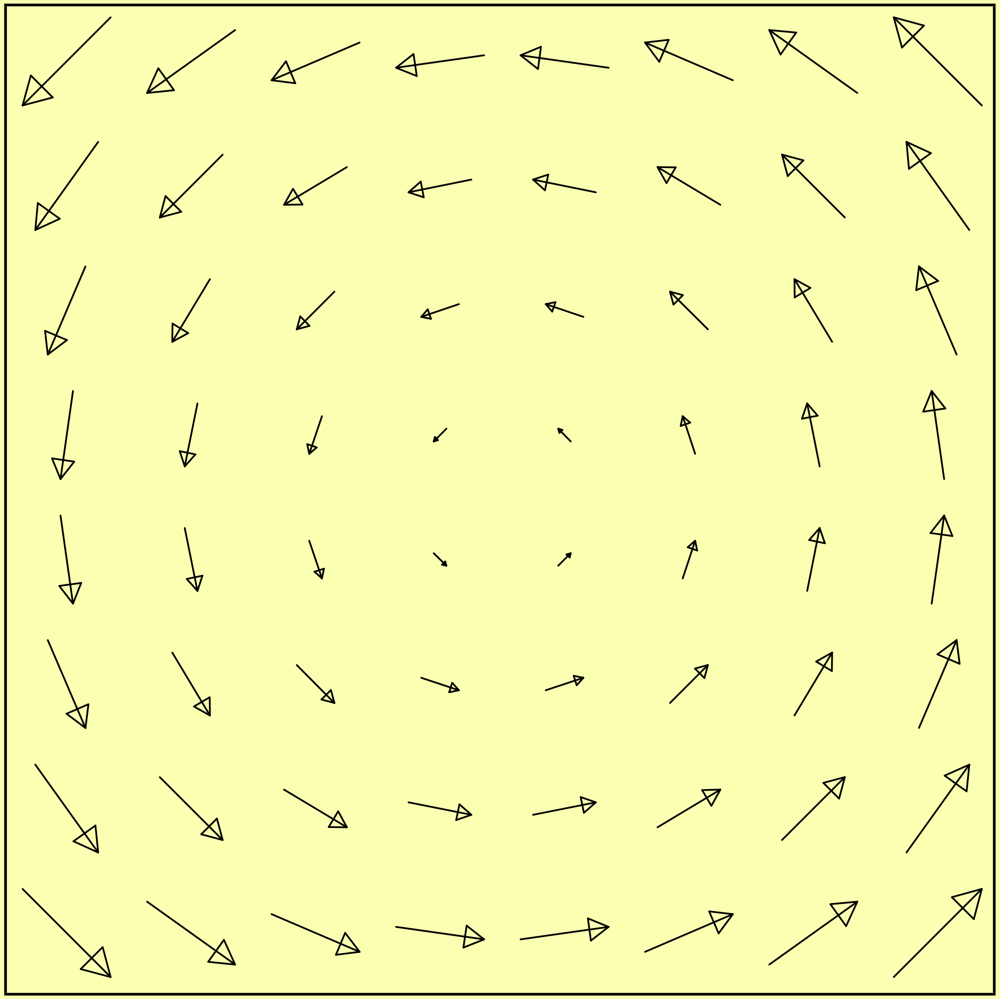


THE GEOMETRY OF CURL

$$\oint_{\text{loop}} \vec{\boldsymbol{F}} \cdot d\vec{\boldsymbol{r}} = \int_{\text{inside}} (\vec{\nabla} \times \vec{\boldsymbol{F}}) \cdot d\vec{\boldsymbol{A}}$$

$$(\vec{\nabla} \times \vec{\boldsymbol{F}}) \cdot \hat{\boldsymbol{n}} \approx \frac{\oint \vec{\boldsymbol{F}} \cdot d\vec{\boldsymbol{r}}}{\text{area of loop}} = \frac{\text{(oriented) circulation}}{\text{unit area}}$$

THE GEOMETRY OF CURL



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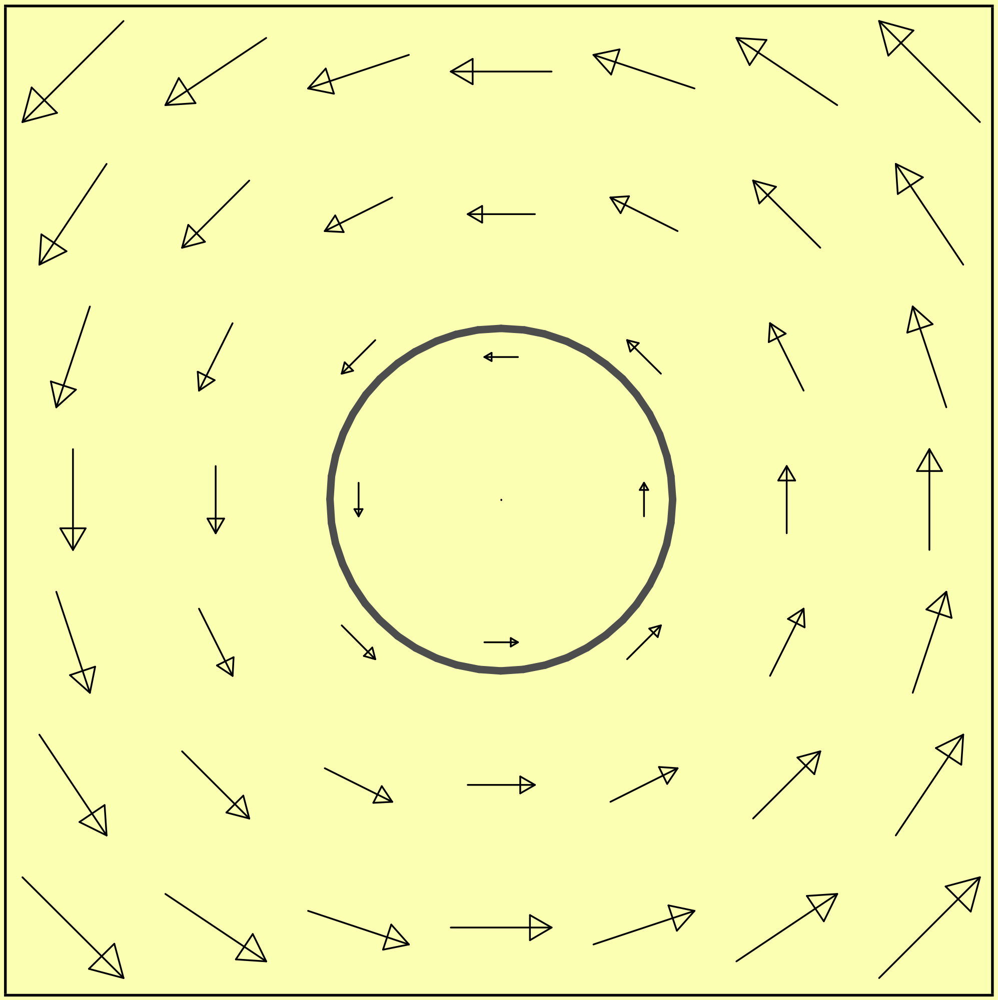

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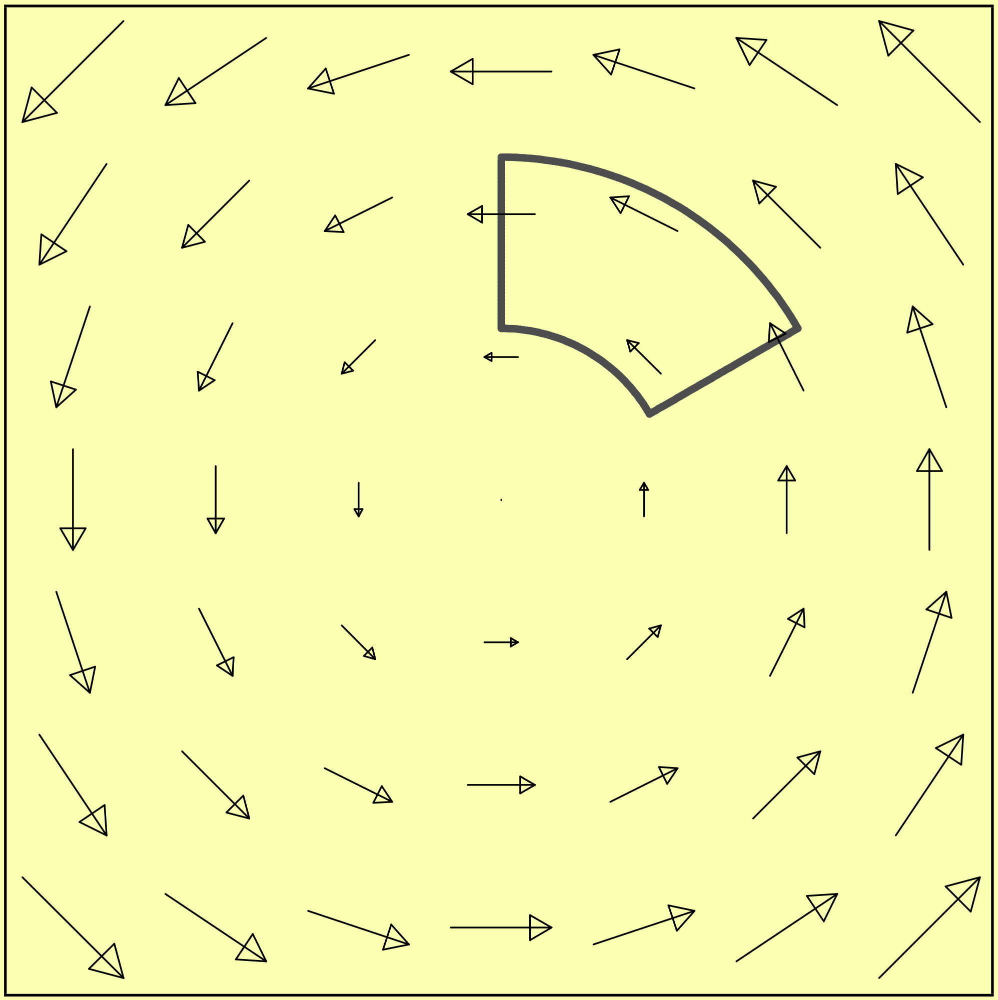

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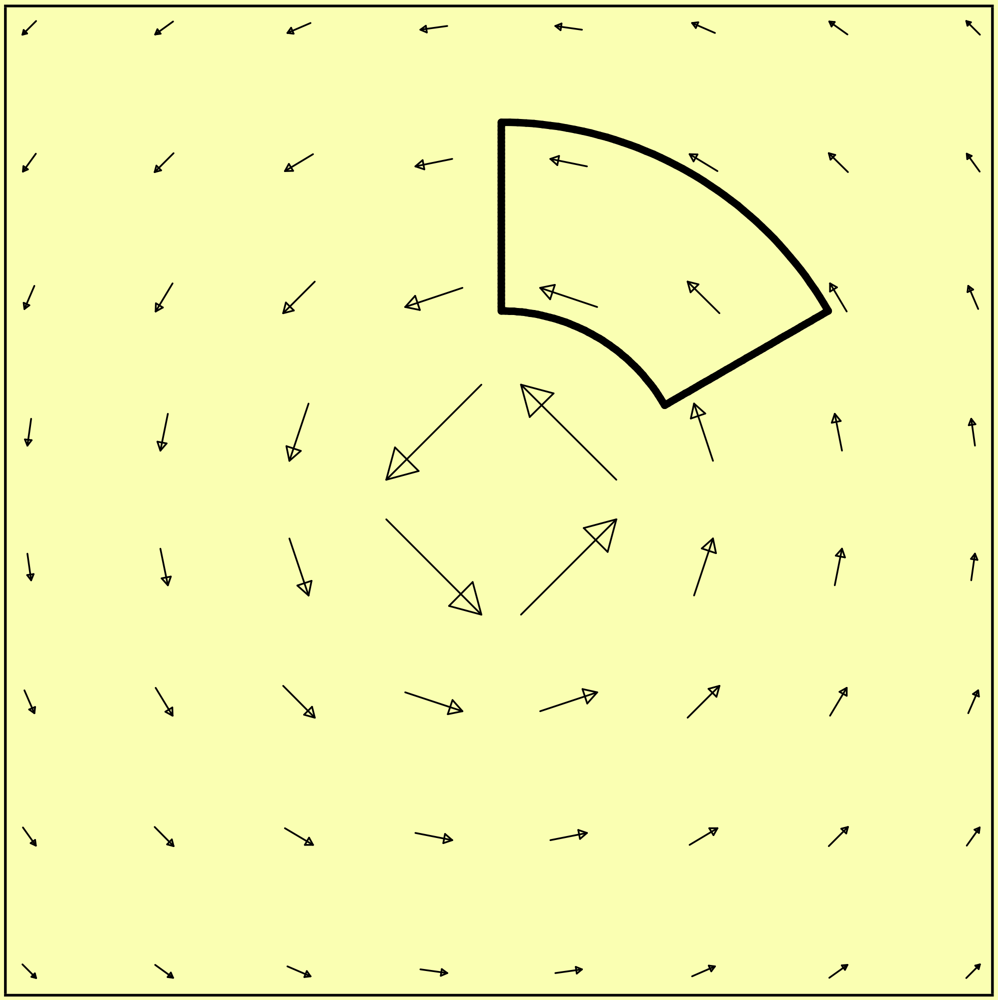

THE GEOMETRY OF CONSERVATIVE VECTOR FIELDS

$$\vec{\boldsymbol{F}} = \vec{\nabla} f$$

$$\vec{\boldsymbol{F}} \cdot d\vec{\boldsymbol{r}} = \vec{\nabla} f \cdot d\vec{\boldsymbol{r}} = df$$

$$\oint \vec{\boldsymbol{F}} \cdot d\vec{\boldsymbol{r}} = 0$$

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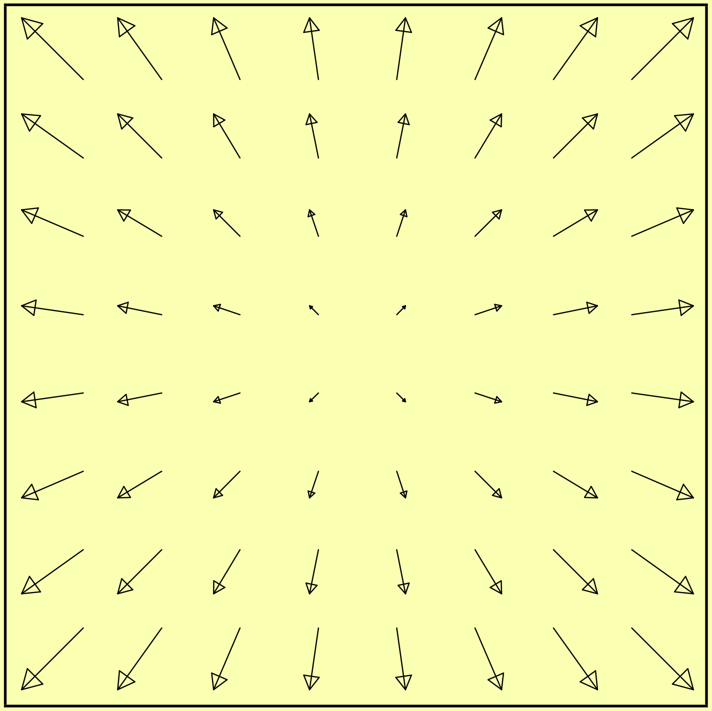

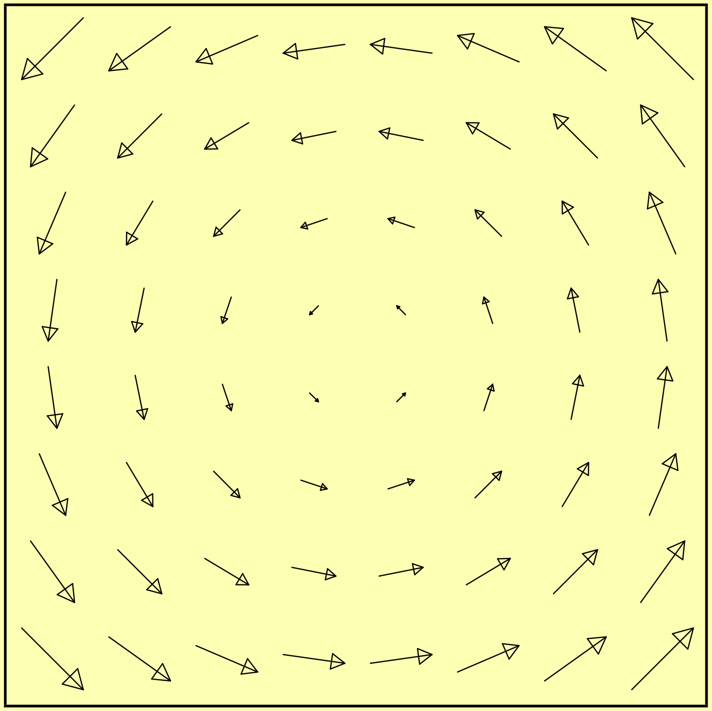

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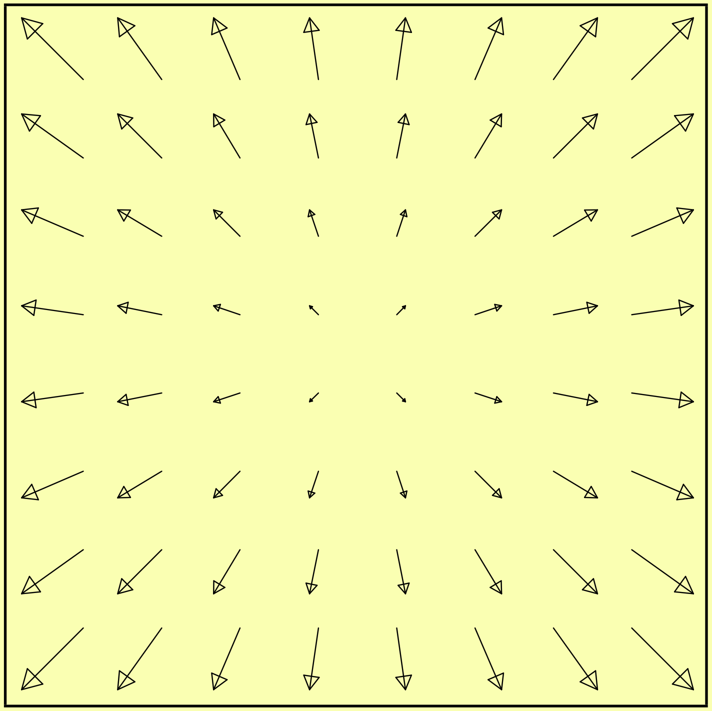

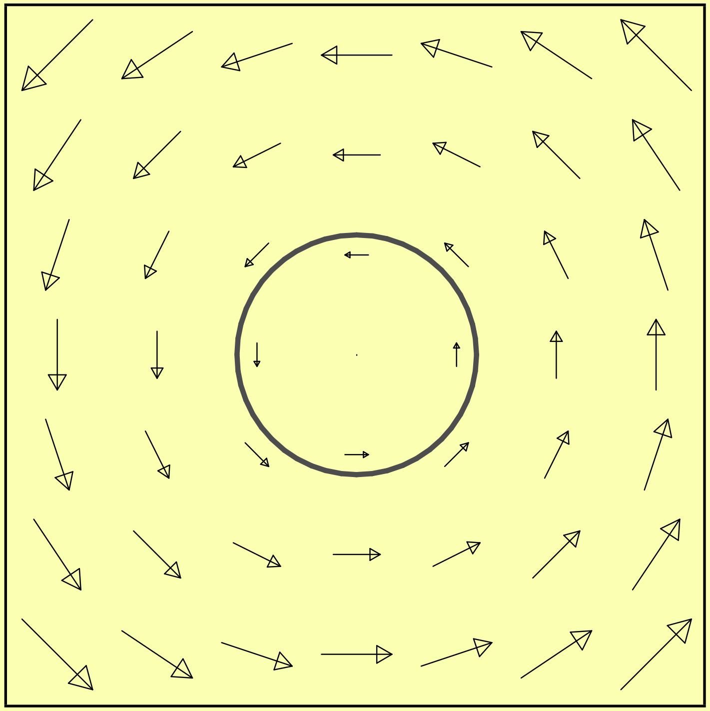

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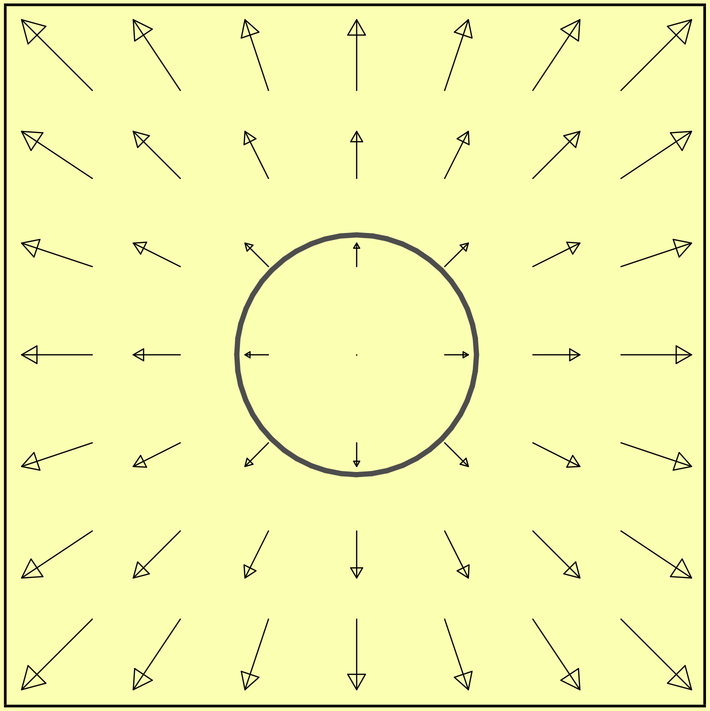

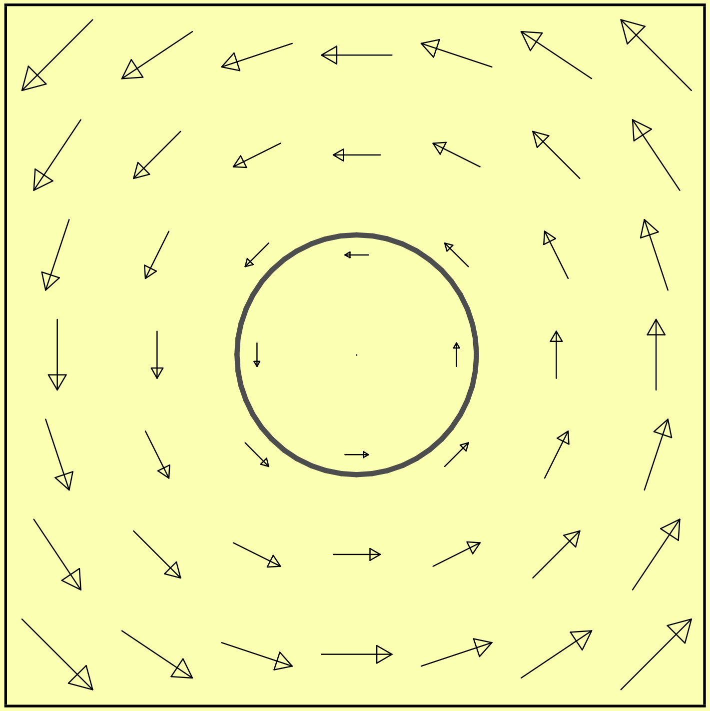

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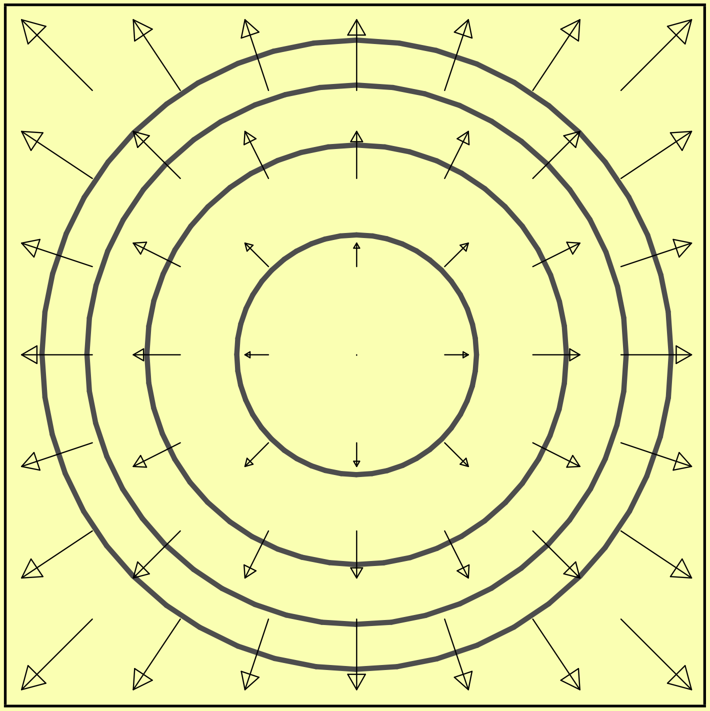

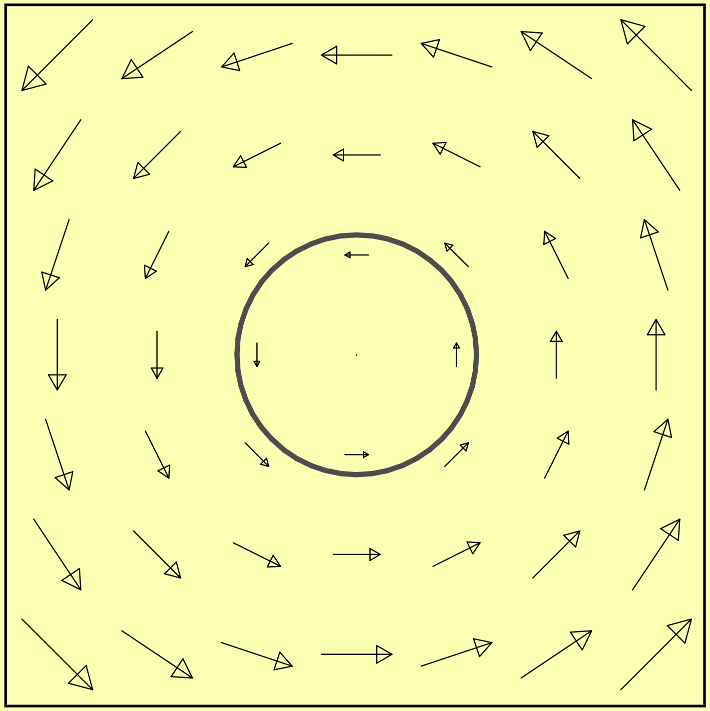

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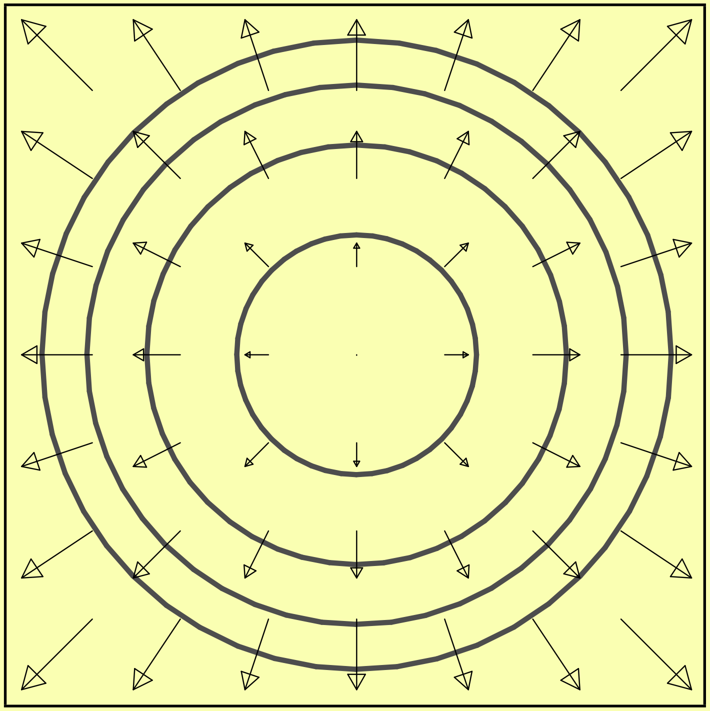

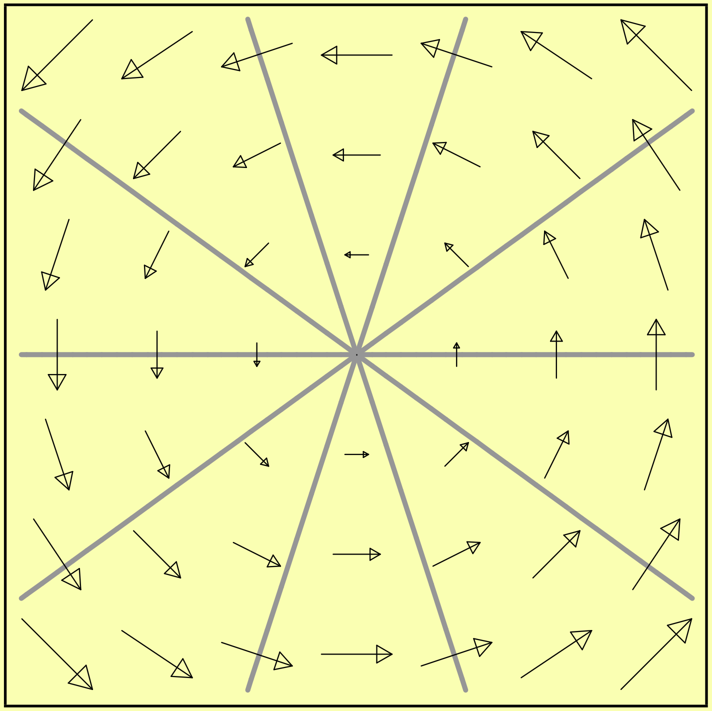

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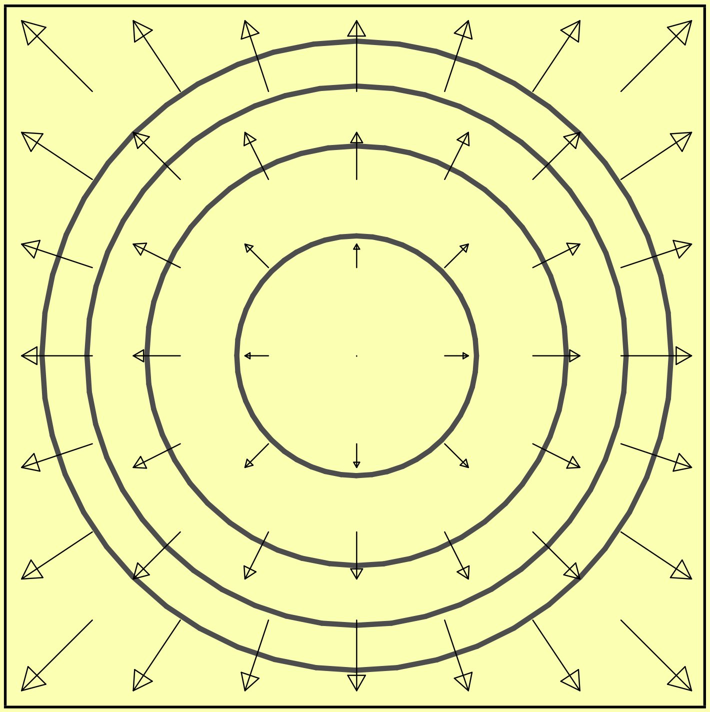

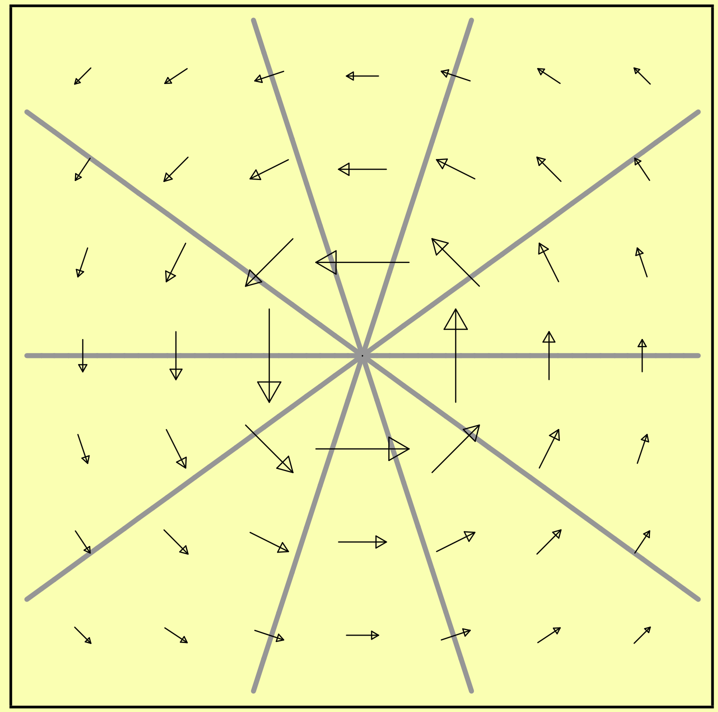

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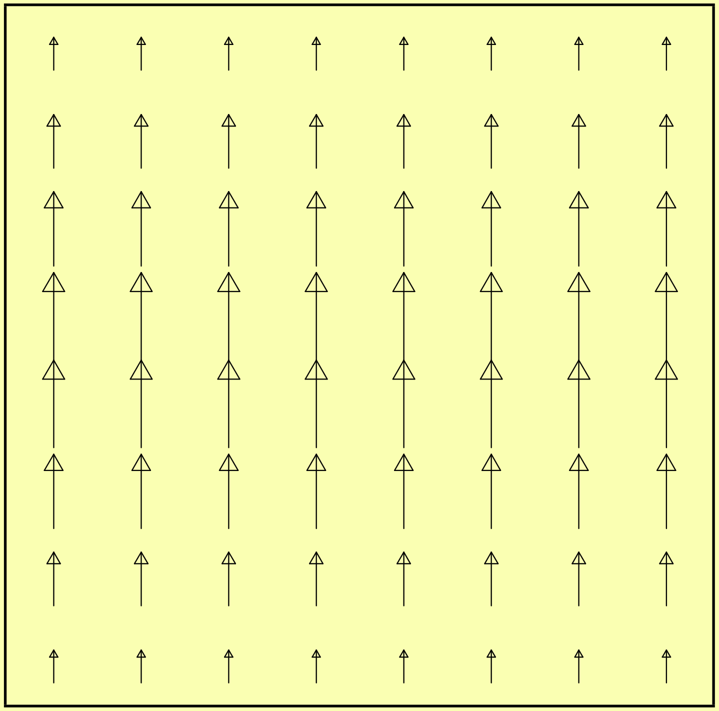

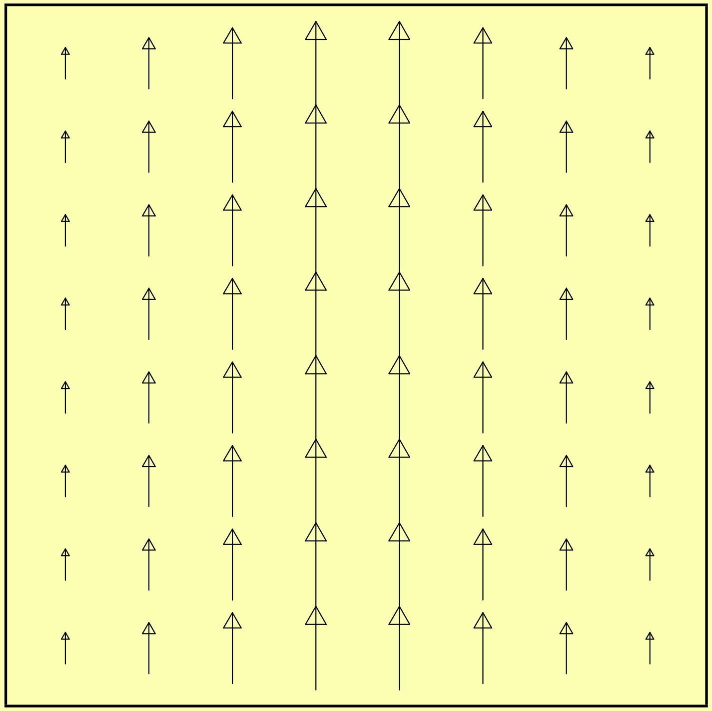

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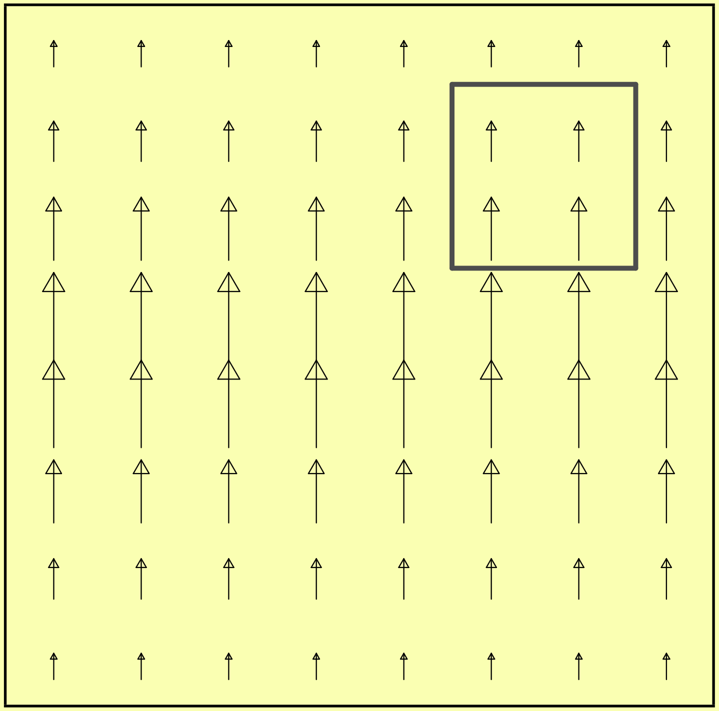

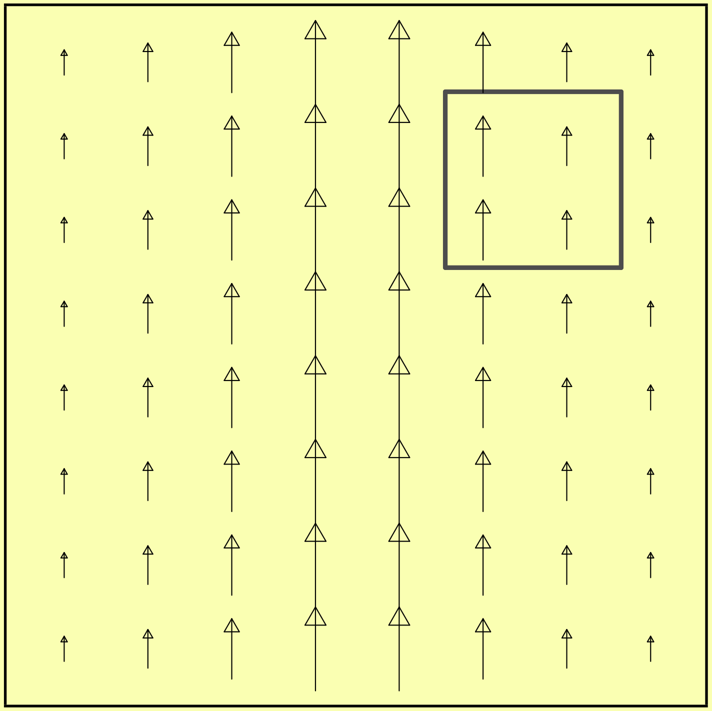

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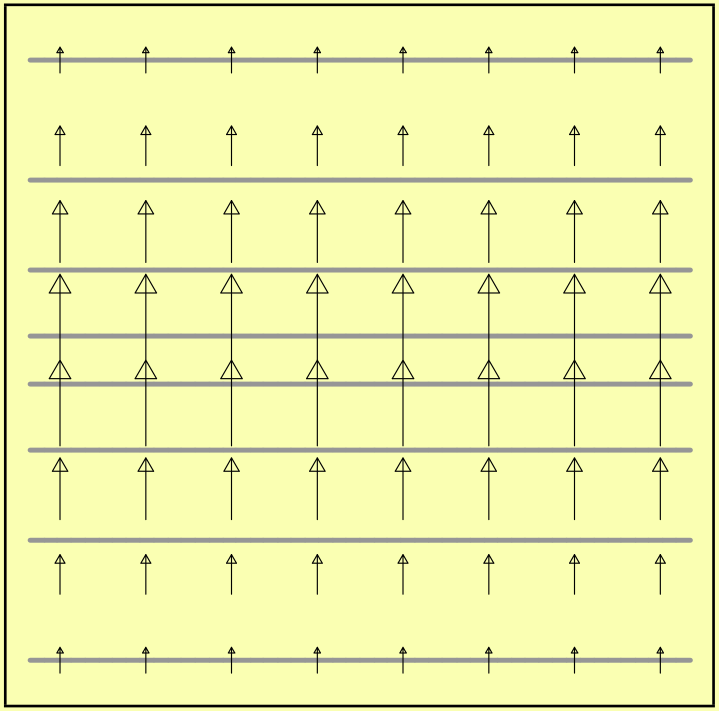

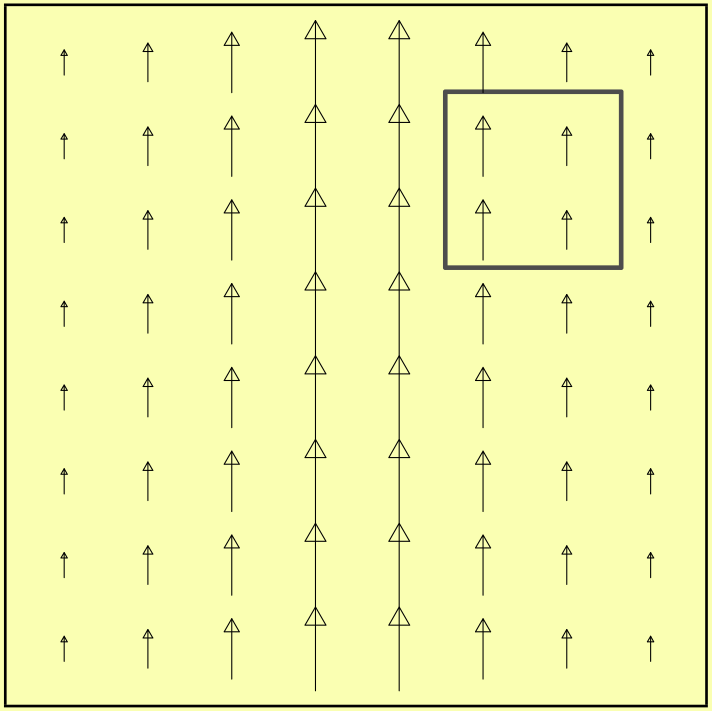

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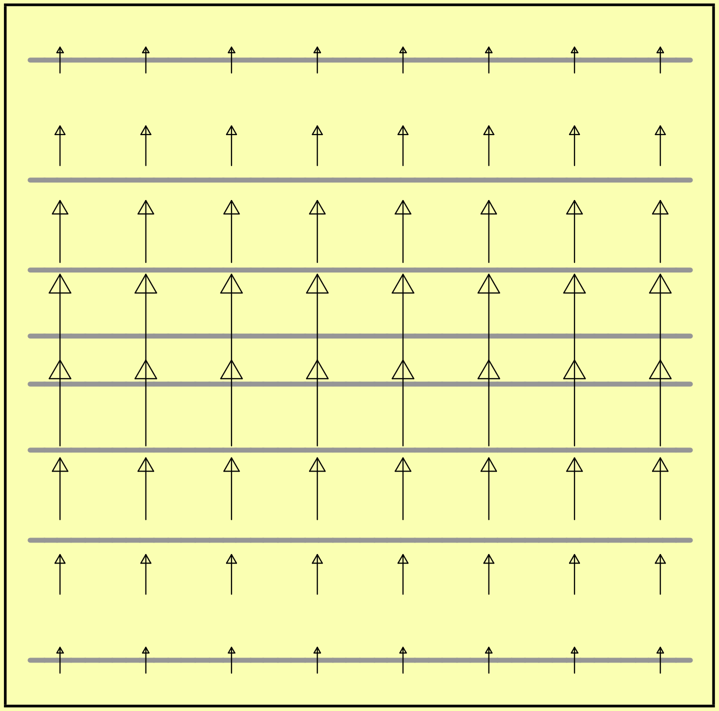

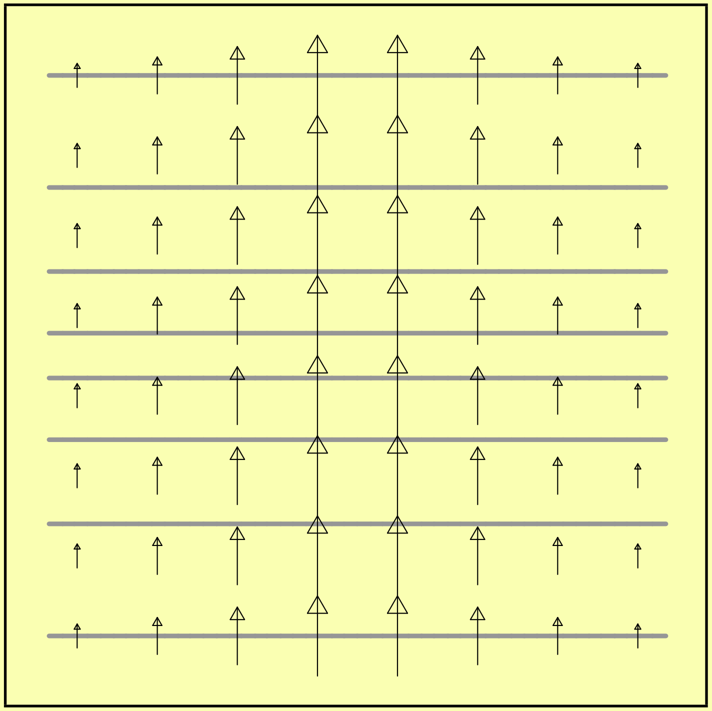

SUMMARY

Geometric visualization
is the key to bridging the gap
between mathematics
and the physical sciences