Bridging the G A P between Mathematics and the Physical Sciences

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http://www.math.oregonstate.edu/bridge

Support

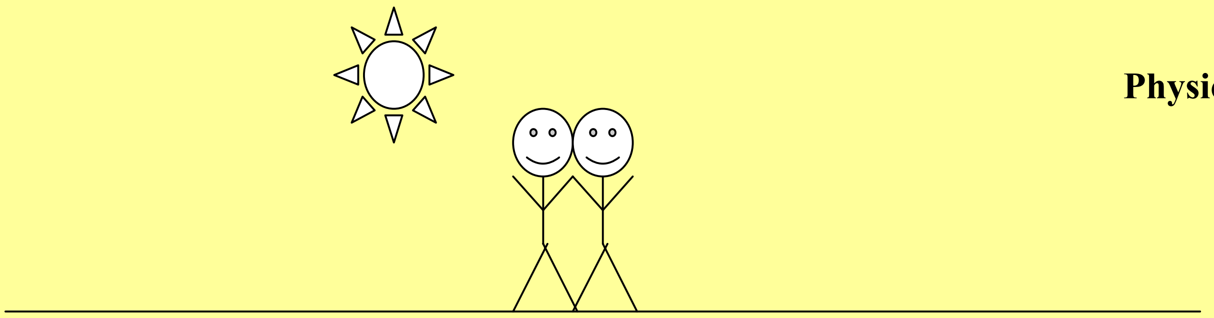
- Mathematical Association of America
  - Professional Enhancement Program
- Oregon State University
  - Department of Mathematics
  - Department of Physics
- National Science Foundation
  - DUE-9653250
  - DUE-0231032
  - DUE-0088901
  - DUE-0231194

Support

- **Grinnell College**
  - **Noyce Visiting Professorship**
- **Mount Holyoke College**
  - **Hutchcroft Fund**
- **Oregon Collaborative for Excellence in the Preparation of Teachers**

Mathematics Physics

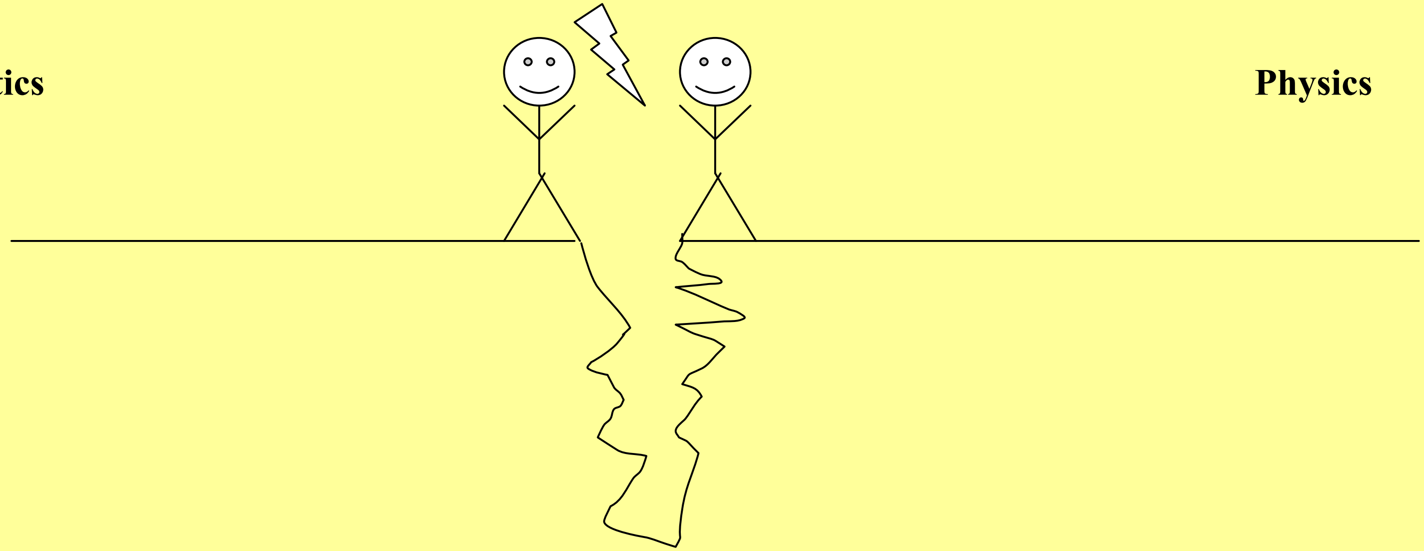
Mathematics



Physics

The Grand Canyon

Mathematics



Physics

What Are Functions?

$$T(x, y) = k(x^2 + y^2)$$

$$T(r, \theta) = ?$$

What Are Functions?

Suppose the temperature on a rectangular slab of metal is given by:

$$T(x, y) = k(x^2 + y^2)$$

Physics: $T(r, \theta) = k\, r^2$

Math: $T(r, \theta) = k(r^2 + \theta^2)$

What Are Functions?

Math:

$$T = f(x, y) = k(x^2 + y^2)$$

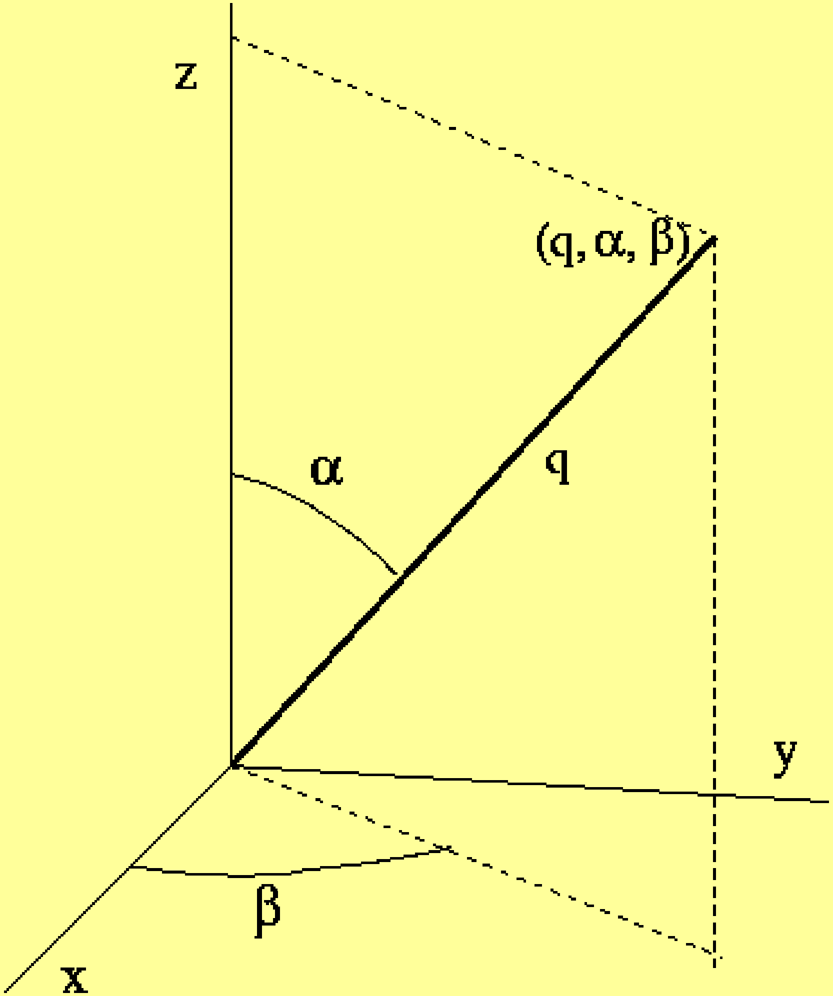
$$T = g(r, \theta) = k\,r^2$$

Physics:

$$T = T(x, y) = k(x^2 + y^2)$$

$$T = T(r, \theta) = k\,r^2$$

Spherical Coordinates



q: Radius

α: Zenith (colatitude)

β: Azimuth (longitude)

Math: (ρ, ϕ, θ)

Physics: (r, θ, ϕ)

The Bridge Project

- Small group activities
- Instructor's Guide
- Study Guide
- CWU, LBCC, MHC, OSU, UPS, UWEC
- Workshops

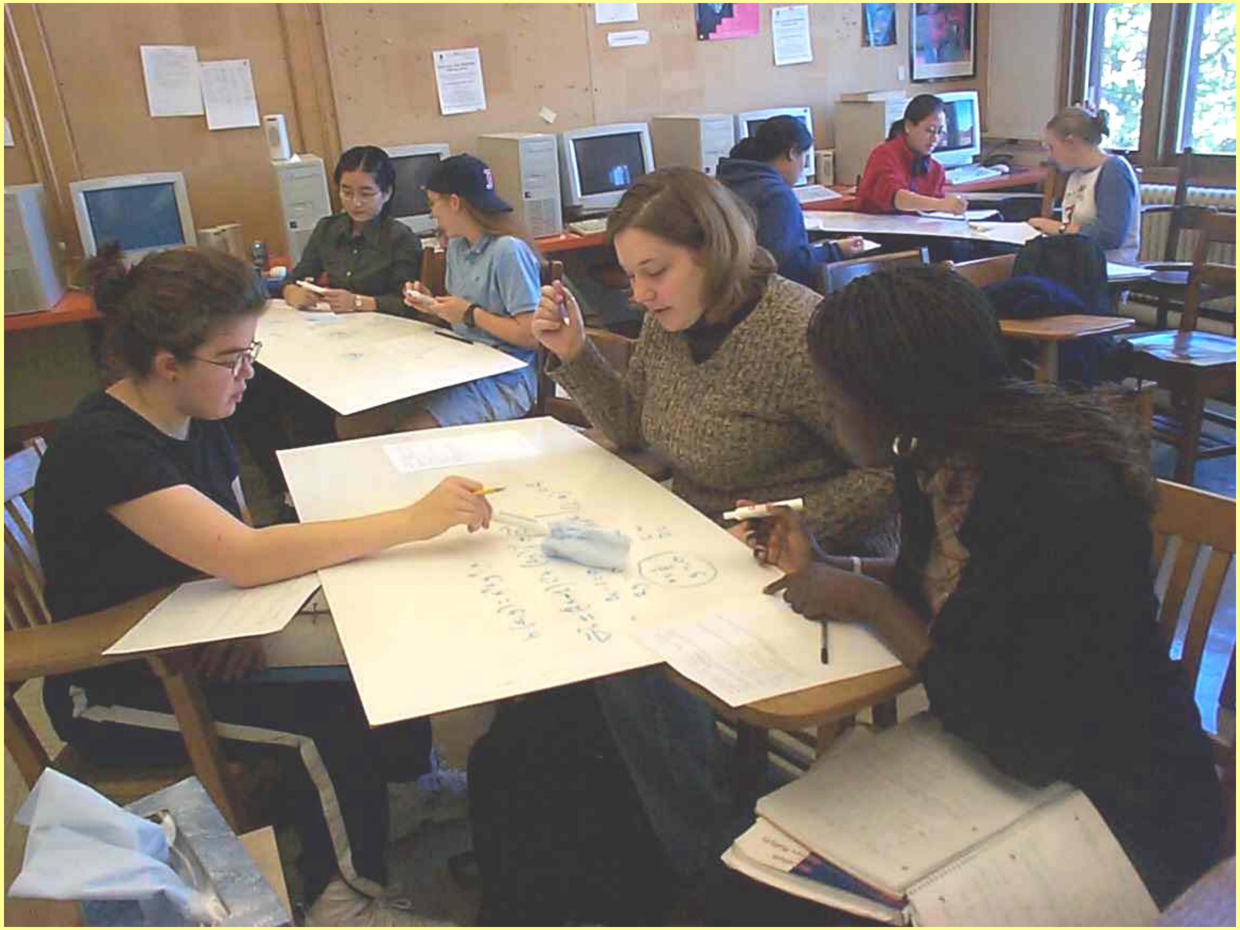
http://www.math.oregonstate.edu/bridge

http://www.physics.oregonstate.edu/bridge

The Bridge Project

- **Differentials**
  - **Use what you know!**
- **Multiple Representations**
- **Symmetry**
  - **Curvilinear coordinates and adapted bases**
- **Geometry**
  - **Vectors; div, grad, curl; …**

Interactive Classroom



Group Activities

- **Task Master**

  Keeps group on track:

  "What you had for lunch doesn't seem relevant."

- **Cynic**

  Questions everything:

  "Why?" "Why?" "Why?"

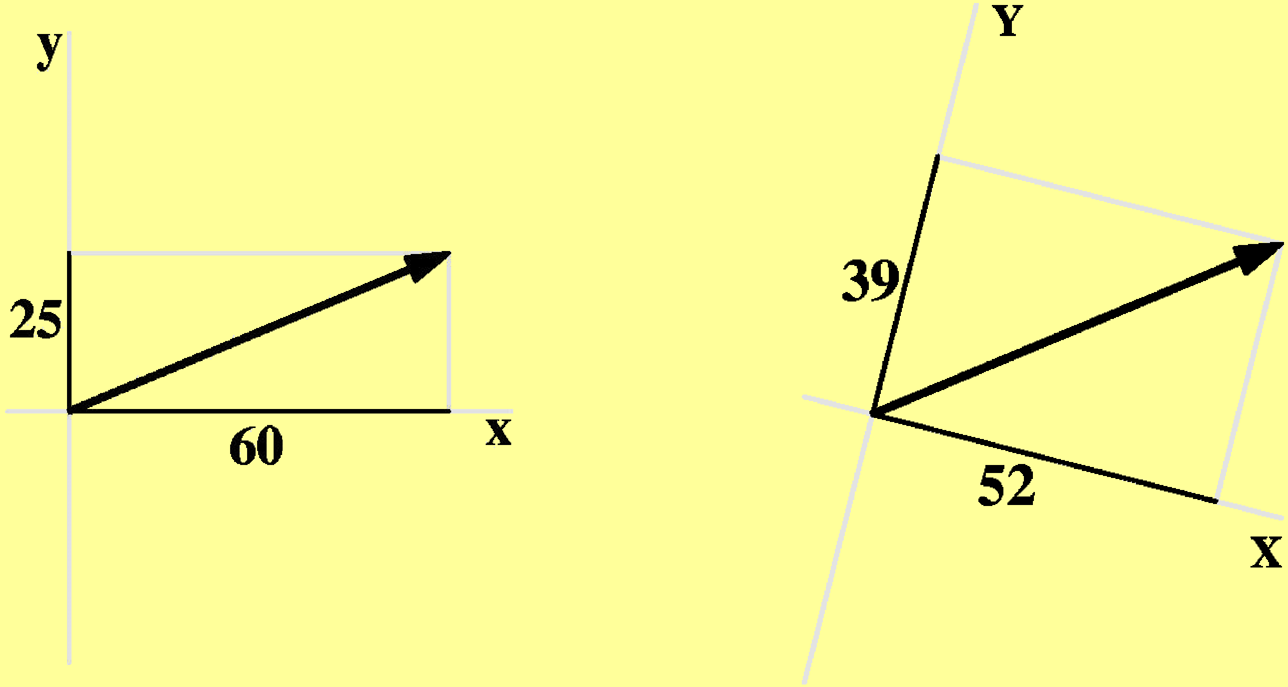
- **Recorder**
- **Reporter**

Coriander

1. In the small town of Coriander, the library can be found by starting at the center of the town square, walking 25 meters north, turning 90° to the right, and walking a further 60 meters.

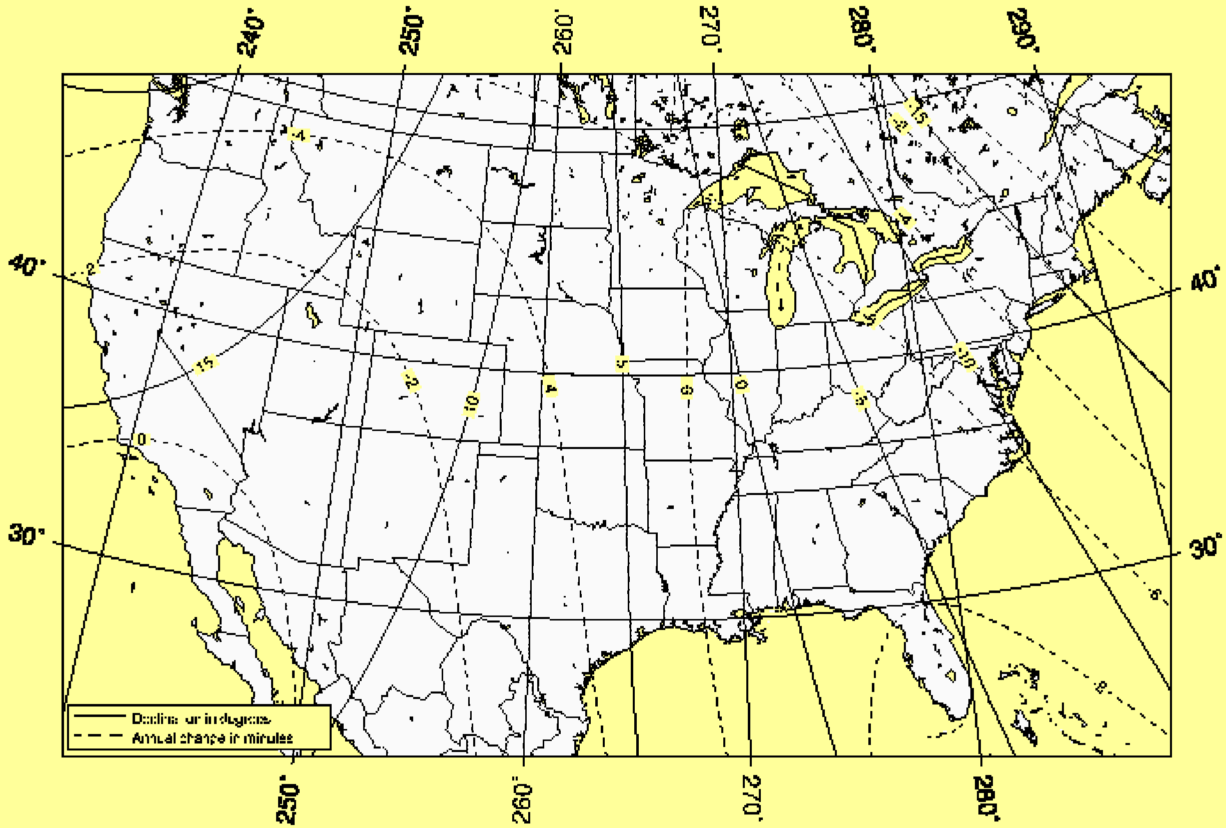
2. Magnetic north in Coriander is approximately 14° east of true north. If you use a compass to find the library (!), the above directions will fail. Instead, you must walk 39 meters in the direction of magnetic north, turn 90° to the right, and walk a further 52 meters.

Coriander



Where on Earth is Coriander?

Coriander



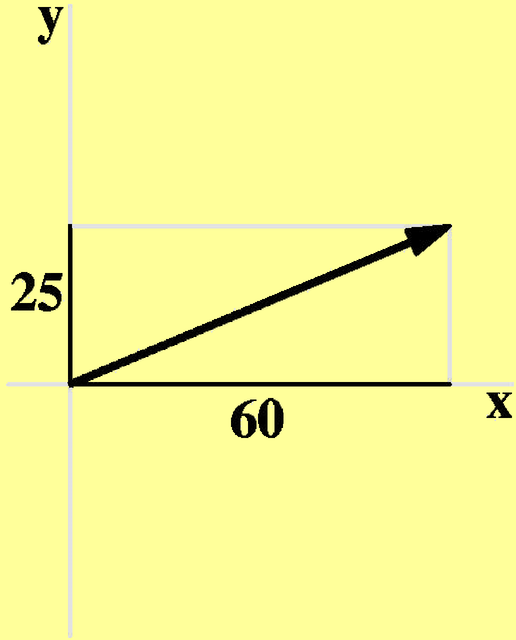
Where on Earth is Coriander?

What Are Vectors?

Mathematics:

- **Triples of numbers**

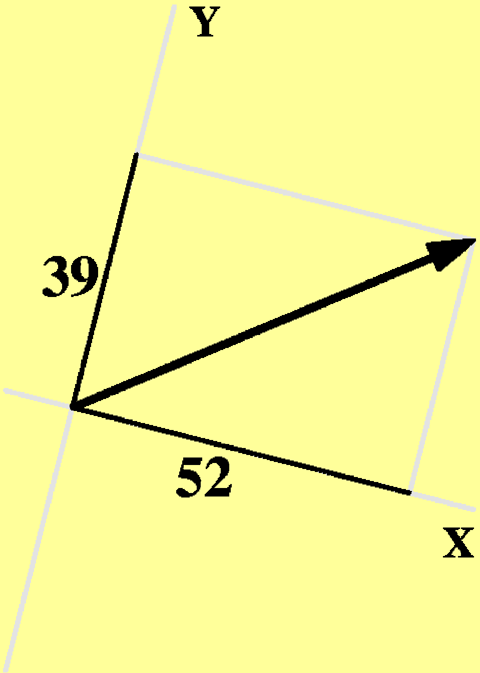
$$\vec{v} = \langle v_1, v_2, v_3 \rangle$$



Physics:

- **Arrows in space**

$$\vec{v} = v_1\hat{\imath} + v_2\hat{\jmath} + v_3\hat{k}$$



Do You Do This?

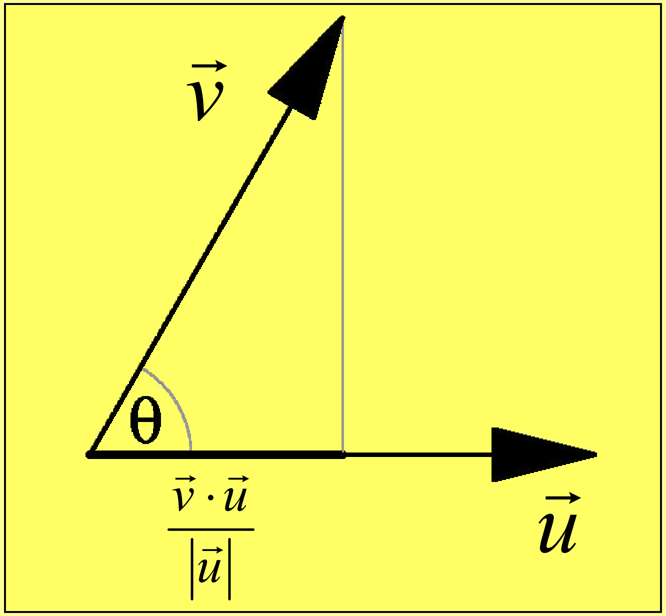
Or This?

$$\vec{u}\cdot\vec{v} = u_1v_1 + u_2v_2 + u_3v_3$$

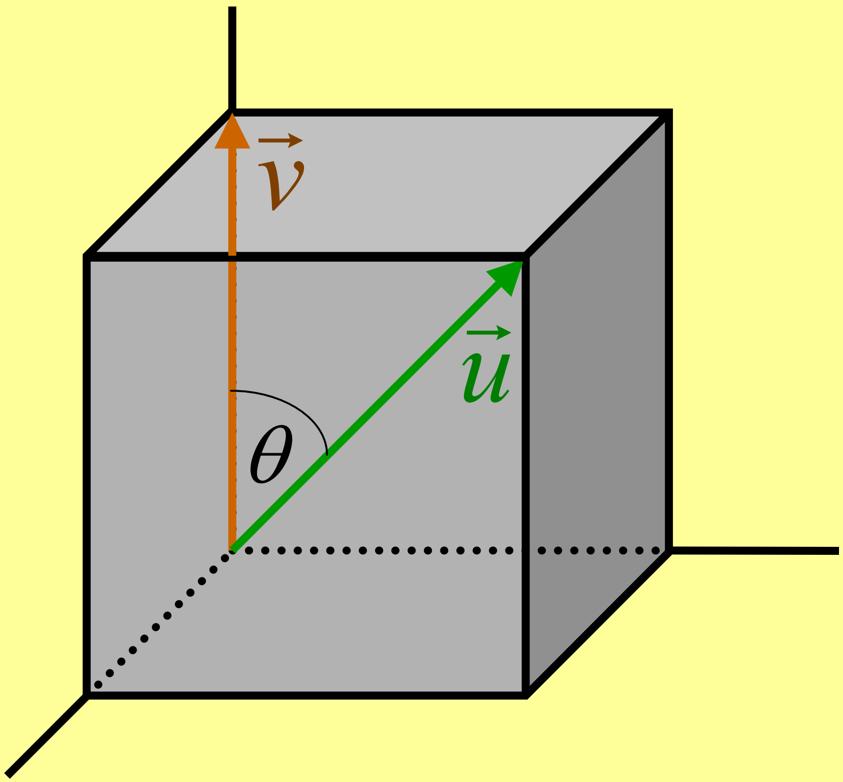
$$\left|\vec{u}\right| = \sqrt{u_1{}^2 + u_2{}^2 + u_3{}^2}$$

$$\vec{u}\cdot\vec{v} = \left|\vec{u}\right|\left|\vec{v}\right|\cos\theta$$

$$\left|\vec{u}\right| = \sqrt{\vec{u}\cdot\vec{u}}$$



The Cube



$$\vec{u} = \hat{\imath} + \hat{\jmath} + \hat{k}$$

$$\vec{v} = \hat{k}$$

$$\cos(\theta) = \frac{\vec{u} \cdot \vec{v}}{|\vec{u}||\vec{v}|} = \frac{1}{\sqrt{3}}$$

Find the angle between a diagonal of a cube and an edge

The Cube

- **Emphasizes that vectors are arrows**
- **Combines geometry and algebra**
- **Uses multiple representations**

geometry: $\vec{u} \cdot \vec{v} = |\vec{u}||\vec{v}|\cos\theta$

algebra: $\vec{u} \cdot \vec{v} = \left(u_1\hat{\imath} + u_2\hat{\jmath} + u_3\hat{k}\right) \cdot \left(v_1\hat{\imath} + v_2\hat{\jmath} + v_3\hat{k}\right)$

memory: $\vec{u} \cdot \vec{v} = u_1v_1 + u_2v_2 + u_3v_3$

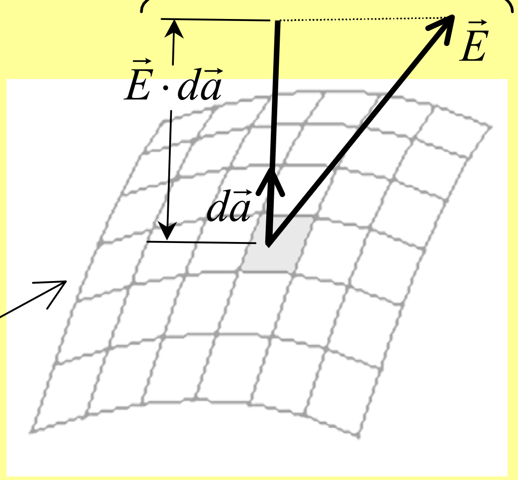
Multiple Representations

Flux is the total amount of electric field through a given area.

Perpendicular Component

Normal Vector

$$\Phi = \int \vec{E} \cdot d\vec{a}$$



$\sum$ over all rectangles

MAA Resources

- **CUPM**

  (MAA Committee on the Undergraduate Program in Mathematics

  - **Curriculum Guide**

    `http://www.maa.org/cupm/cupm2004.pdf`

- **CRAFTY**

  (Subcommittee on Curriculum Renewal Across the First Two Years)

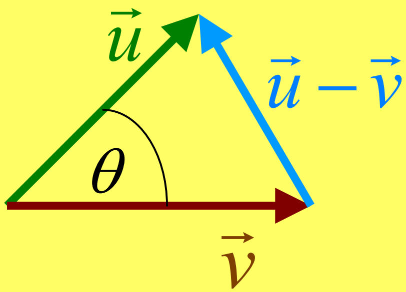
  - **Voices of the Partner Disciplines**

    `http://www.maa.org/cupm/crafty`

Start with Algebra?

$$\vec{u}\cdot\vec{v} = u_1v_1 + u_2v_2 + u_3v_3$$

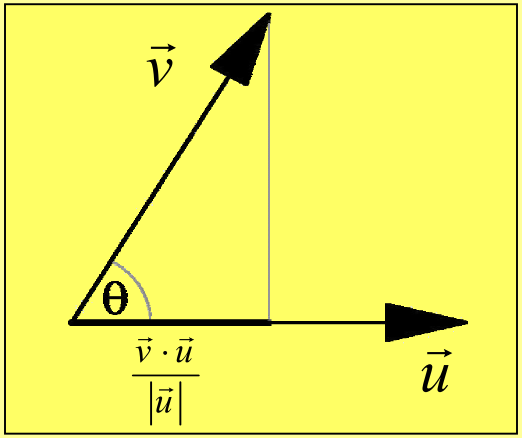
$$\Rightarrow (\vec{u}-\vec{v})\cdot(\vec{u}-\vec{v}) = \vec{u}\cdot\vec{u} + \vec{v}\cdot\vec{v} - 2\vec{u}\cdot\vec{v}$$

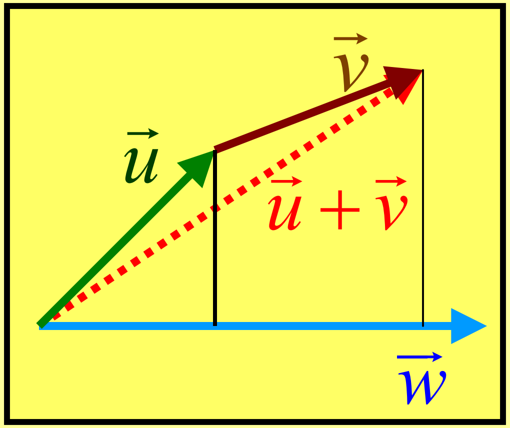


(Law of Cosines)

$$\left|\vec{u}-\vec{v}\right|^2 = \left|\vec{u}\right|^2 + \left|\vec{v}\right|^2 - 2\left|\vec{u}\right|\left|\vec{v}\right|\cos\theta$$

Start with Geometry!



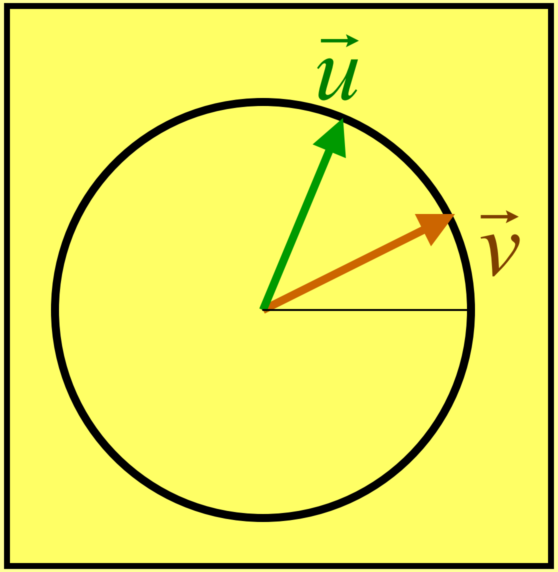


$$\Rightarrow (\vec{u}+\vec{v})\cdot\vec{w} = \vec{u}\cdot\vec{w} + \vec{v}\cdot\vec{w}$$

$$\Rightarrow \left(u_1\hat{\imath} + u_2\hat{\jmath}\right)\cdot\left(v_1\hat{\imath} + v_2\hat{\jmath}\right) = u_1v_1 + u_2v_2$$

(get Law of Cosines for free!)

Use both!



$$\vec{u} = \cos(\alpha)\,\hat{\imath} + \sin(\alpha)\,\hat{\jmath}$$

$$\vec{v} = \cos(\beta)\,\hat{\imath} + \sin(\beta)\,\hat{\jmath}$$

$$\vec{u} \cdot \vec{v} = \cos(\alpha - \beta)$$

(get addition formulas for free!)

Differentials

- **Substitution**

$$\int 2x \sin x \, dx$$

$$u = x^2$$

$$du = 2x \, dx$$

- **Chain Rule**

$$x = \cos\theta$$

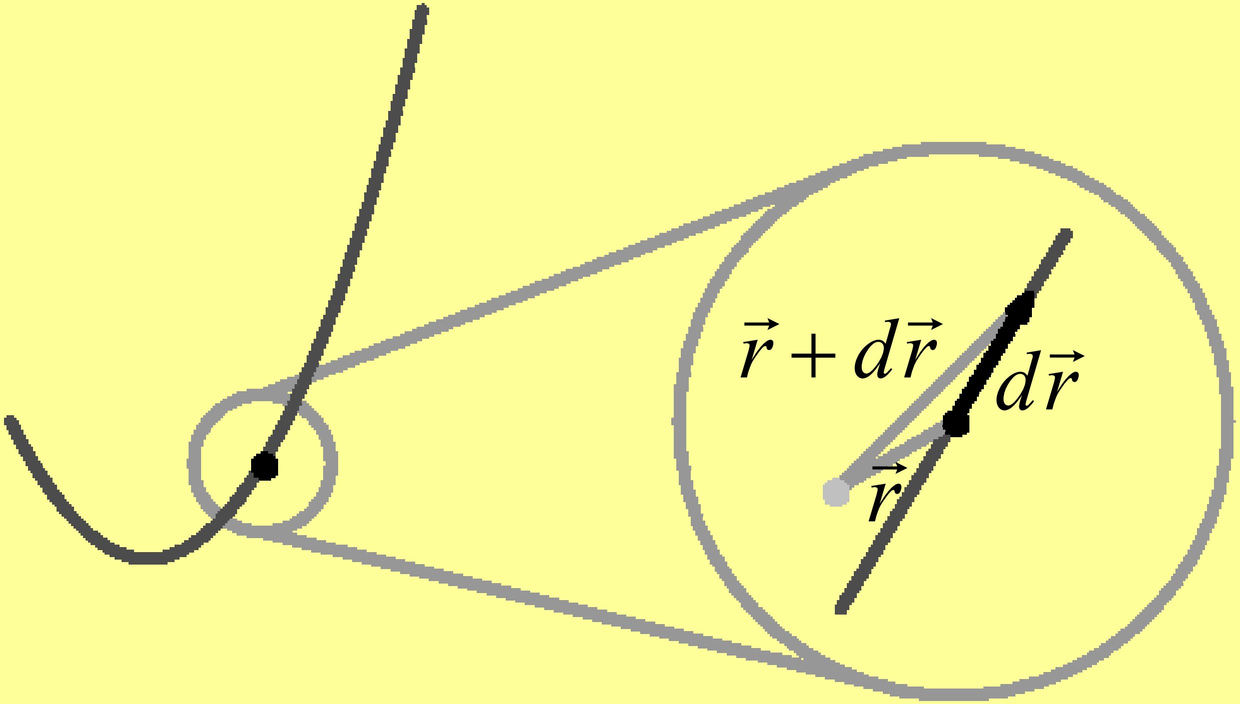
$$dx = -\sin\theta \, d\theta$$

$$u = x^2$$

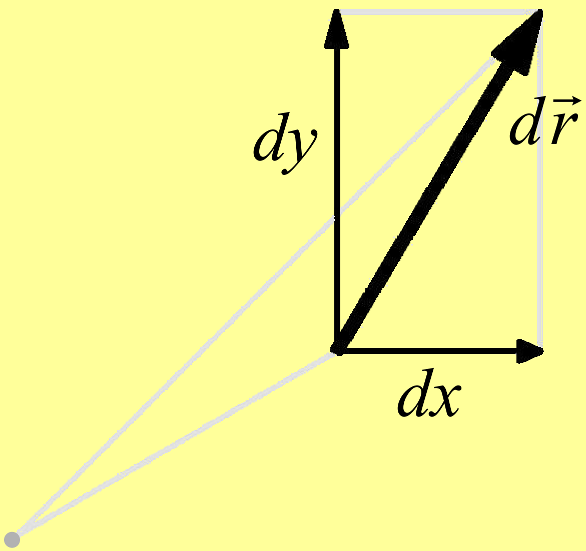
$$du = 2x \, dx$$

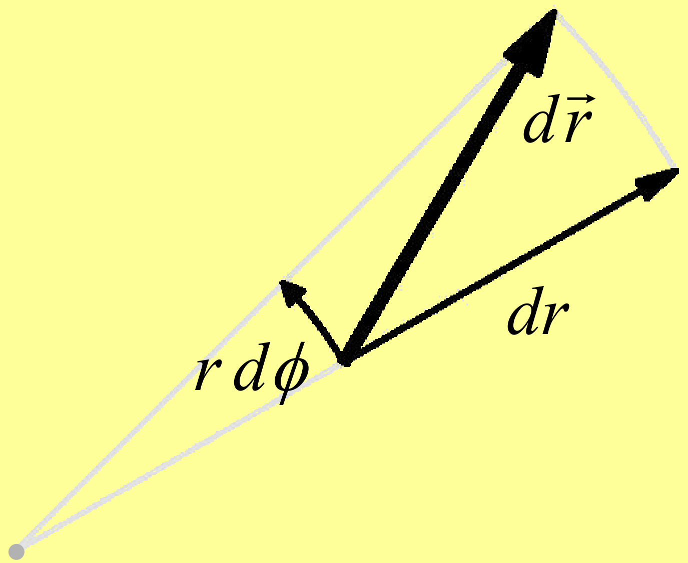
$$= 2\cos\theta(-\sin\theta \, d\theta)$$

Vector Differentials



Vector Differentials





$$d\vec{r} = dx\,\hat{\imath} + dy\,\hat{\jmath}$$

$$d\vec{r} = dr\,\hat{r} + r\,d\phi\,\hat{\phi}$$

Vector Differentials

- **Line Integrals**

$$\int \vec{F} \cdot d\vec{r}$$

$$\int f\, ds \qquad ds = \left| d\vec{r} \right|$$

- **Surface Integrals**

$$\int\int \vec{F} \cdot d\vec{S} \qquad d\vec{S} = d\vec{r}_1 \times d\vec{r}_2$$

$$\int\int f\, dS \qquad dS = \left| d\vec{r}_1 \times d\vec{r}_2 \right|$$

- **Volume Integrals**

$$dV = (d\vec{r}_1 \times d\vec{r}_2) \cdot d\vec{r}_3$$

- **Gradient**

$$df = \vec{\nabla} f \cdot d\vec{r}$$

Mathematics $\neq$ Physics

- Physics is about things.
- Physicists can't change the problem.

Physics is about things

- What sort of a beast is it?
  - Scalar fields, units, coordinates, time.
- Physics is independent of coordinates.
  - Vectors as arrows, geometry of dot and cross.
- Graphs are about relationships of physical things.
  - Fundamental physics is 3 dimensional.
  - 3-d graphs of functions of 2-variables are misleading.
  - Hills are not a good example of functions of 2 variables.
  - Use of color.
- Fundamental physics is highly symmetric.
  - Spheres and cylinders vs. parabolas.
  - Interesting physics problems can involve trivial math.
  - Adapted basis vectors, such as $\hat{r},\hat{\theta},\hat{\phi}$.

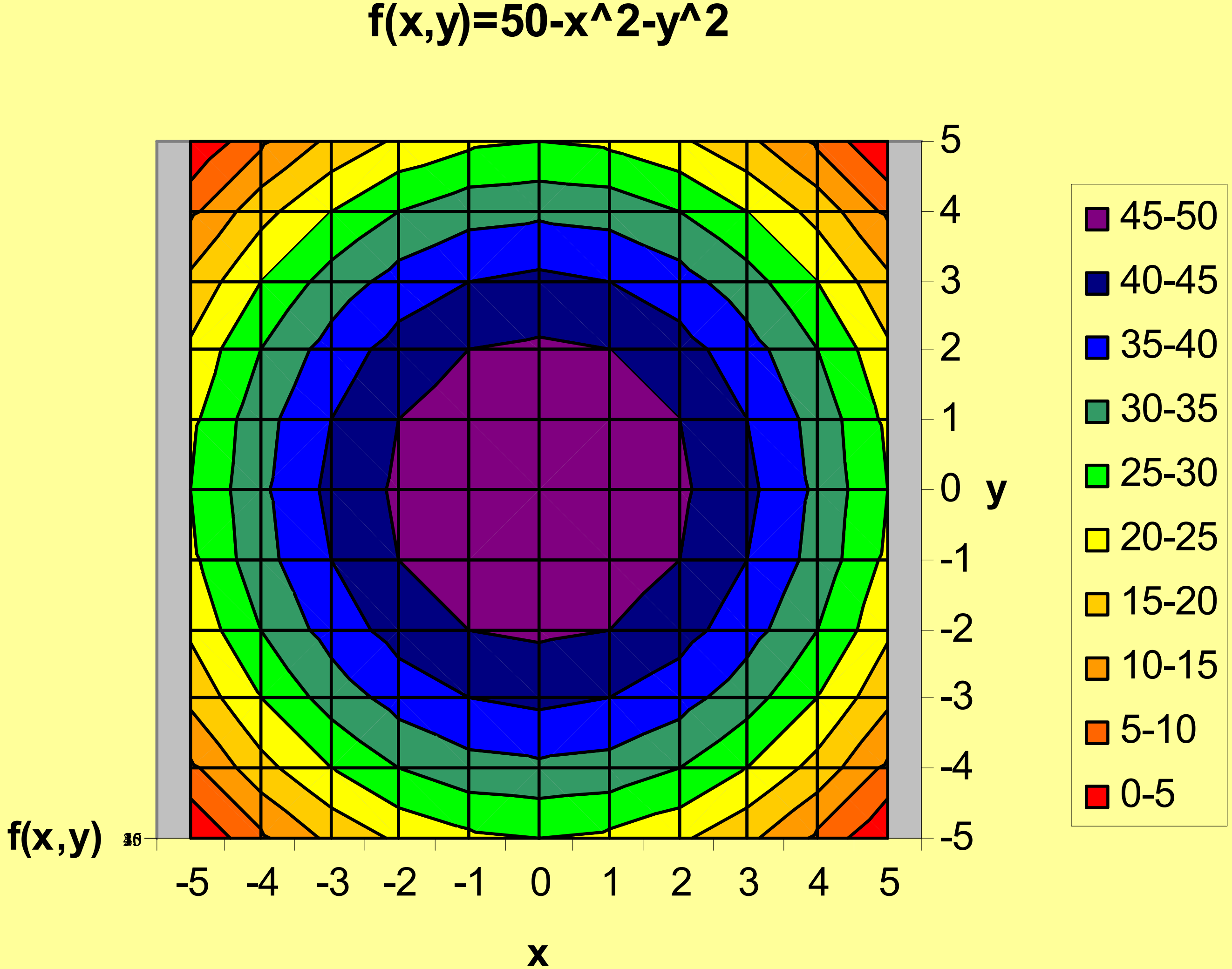
Physicists can't change the problem.

- Physics involves creative synthesis of multiple ideas.
- Physics problems not nec. well-defined math problems.
  - No preferred coordinates or independent variables.
  - No parameterization.
  - Unknowns don't have names.
  - Getting to a well-defined math problem is part of the problem.
  - If you can't add units, it's a poor physics problem.
- Physics problems don't fit templates.
  - Template problem-solving vs. skills.
  - A few key ideas are remembered best later.
- Physics involves interplay of multiple representations.
  - Dot product.

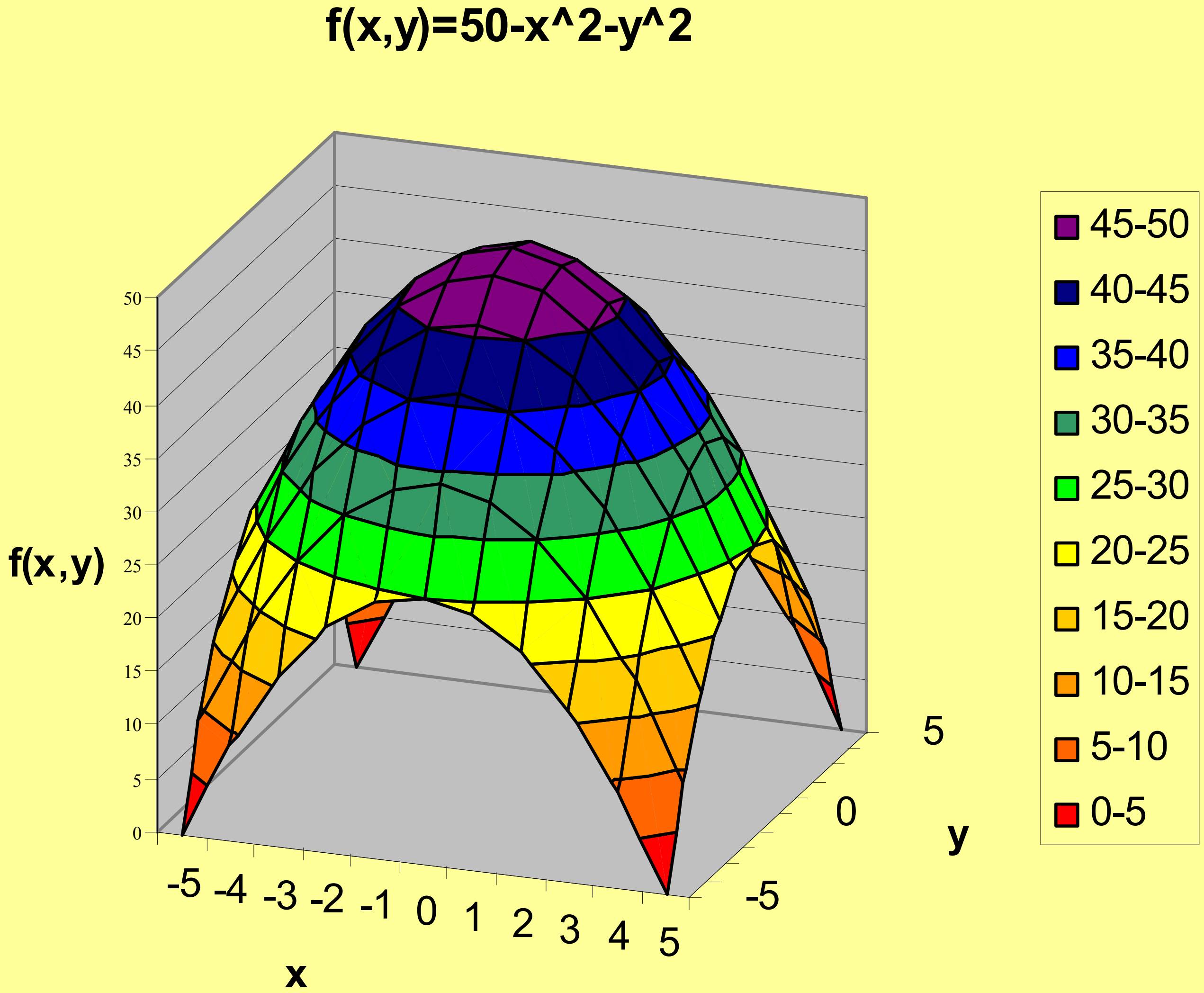
Visualization—Table

	-5	-4	-3	-2	-1	0	1	2	3	4	5
-5	0	9	16	21	24	25	24	21	16	9	0
-4	9	18	25	30	33	34	33	30	25	18	9
-3	16	25	32	37	40	41	40	37	32	25	16
-2	21	30	37	42	45	46	45	42	37	30	21
-1	24	33	40	45	48	49	48	45	40	33	24
0	25	34	41	46	49	50	49	46	41	34	25
1	24	33	40	45	48	49	48	45	40	33	24
2	21	30	37	42	45	46	45	42	37	30	21
3	16	25	32	37	40	41	40	37	32	25	16
4	9	18	25	30	33	34	33	30	25	18	9
5	0	9	16	21	24	25	24	21	16	9	0

Visualization—Level Curves



Visualization—Graph



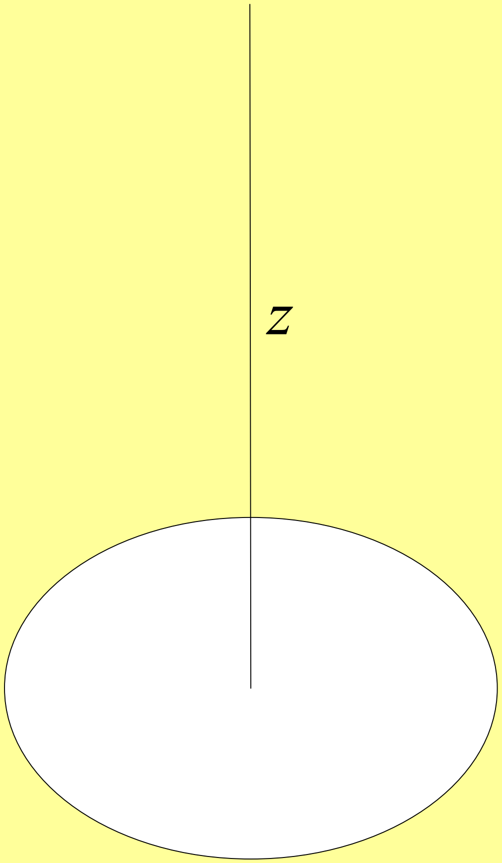
Goals

- Mathematics
  - Visualization
  - Arbitrary surfaces
  - 2-d primary
  - Sophistication now
  - Domain is important
  - End of course flexible
- Physics
  - Visualization
  - Simple surfaces
  - 3-d primary
  - Sophistication later
  - Surface is important
  - End of course fixed

An Example

- **Typical of EARLY upper-division work for physics majors and many engineers.**
- **Solution requires:**
  - **many mathematical strategies,**
  - **many geometrical and visualization strategies,**
  - **only one physics concept.**
- **Demonstrates different use of language.**

Potential Due to Charged Disk



What is the electrostatic potential at a point, on axis, above a uniformly charged disk?

One Physics Concept

- **Coulomb's Law:**

$$V = \frac{1}{4\pi\varepsilon_0}\frac{q}{r}$$

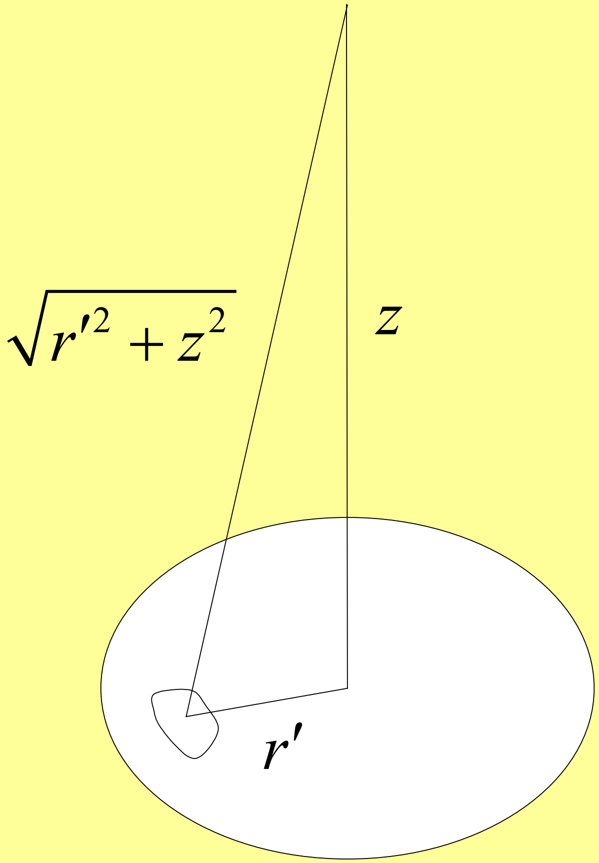
Superposition

- **Superposition for solutions of linear differential equations:**

$$V = \frac{1}{4\pi\varepsilon_0}\frac{q}{r}$$

$$\rightarrow V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int \frac{\sigma(\vec{r}')da'}{\left|\vec{r}-\vec{r}'\right|}$$

Chopping and Adding



Integrals involve chopping up a part of space and adding up a physical quantity on each piece.

Computational Skill

- **Can the students set-up and do the integral?**

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int \frac{\sigma(\vec{r}')da'}{|\vec{r}-\vec{r}'|}$$

$$= \frac{\sigma}{4\pi\varepsilon_0}\int_0^{2\pi}\int_0^{R} \frac{dr'\, r'\, d\theta'}{\sqrt{r'^2+z^2}}$$

$$= \frac{2\pi\sigma}{4\pi\varepsilon_0}\left(\sqrt{R^2+z^2}-z\right)$$

Limits (Far Away)

$$V(\vec{r}) = \frac{2\pi\sigma}{4\pi\varepsilon_0}\left(\sqrt{R^2 + z^2} - z\right)$$

$$= \frac{2\pi\sigma}{4\pi\varepsilon_0}\left(z\sqrt{1 + \frac{R^2}{z^2}} - z\right)$$

$$= \frac{2\pi\sigma}{4\pi\varepsilon_0}\left(z\left(1 + \frac{1}{2}\frac{R^2}{z^2} + \ldots\right) - z\right)$$

$$\approx \frac{1}{4\pi\varepsilon_0}\frac{\pi R^2\sigma}{z}$$

Simple Curriculum Additions

- Equations describe the relationship of physical quantities to other physical quantities.
- Emphasize superposition of solutions to linear differential equations.
- Integrals involve chopping a part of space and adding up a physical quantity on each piece.
- “Limits” require one to say what dimensionless quantity is small.

A Radical View of Calculus

- The central idea of calculus is not the limit.
- The central idea of derivatives is not slope.
- The central idea of integrals is not area.
- The central idea of curves and surfaces is not parameterization.
- The central representation of a function is not its graph.

A Radical View of Calculus

- The central idea of calculus is the differential.
- The central idea of derivatives is rate of change.
- The central idea of integrals is total amount.
- The central idea of curves and surfaces is "use what you know".
- The central representation of a function is data attached to the domain.

Summary

***Geometric visualization
is the key to bridging the gap
between mathematics
and the physical sciences.***