# BRIDGING THE GAP between Mathematics and the Physical Sciences

Tevian Dray & Corinne Manogue
Oregon State University

I: Mathematics $\neq$ Physics

II: The Bridge Project

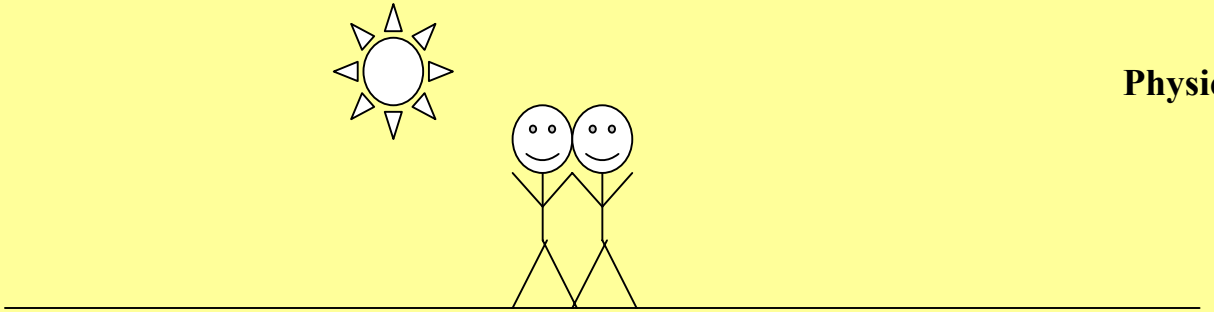
http://www.math.oregonstate.edu/bridge

# Support

- **Oregon State University**
  - **Department of Mathematics**
  - **Department of Physics**
- **Grinnell College**
  - **Noyce Visiting Professorship**
- **Mount Holyoke College**
  - **Hutchcroft Fund**
- **National Science Foundation**
  - **DUE-9653250** – **DUE-0231194**
  - **DUE-0088901** – **DUE-0231032**
- **Oregon Collaborative for Excellence in the Preparation of Teachers**

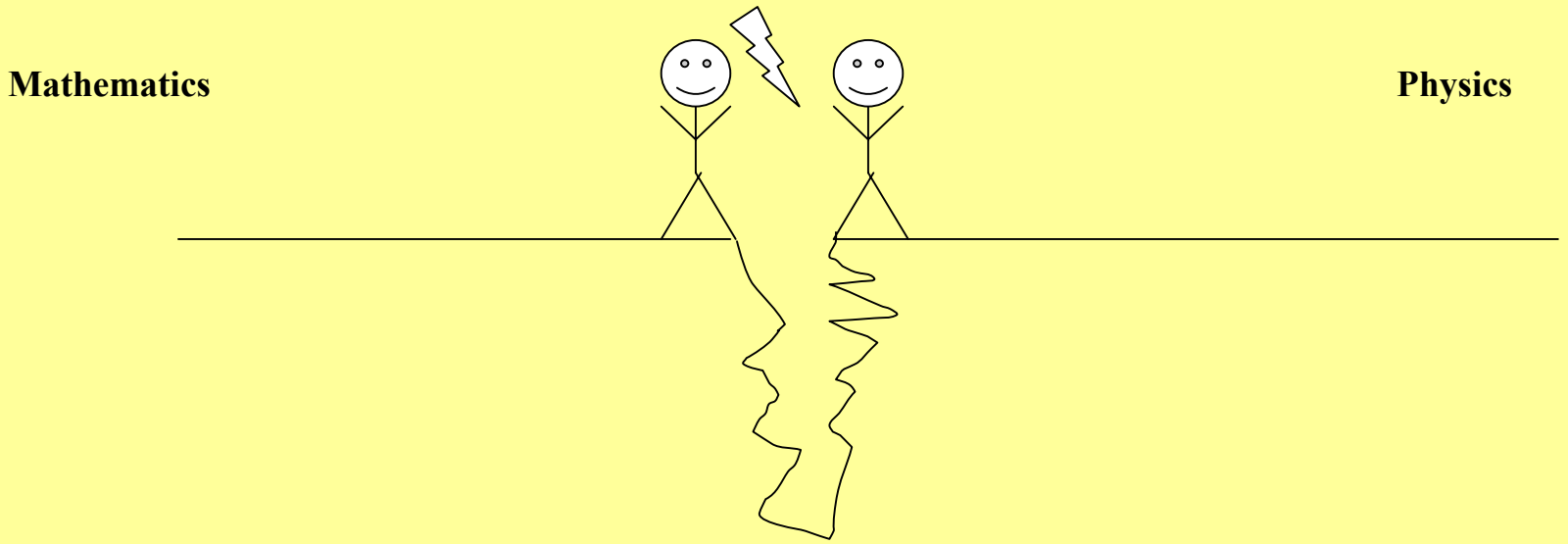
# Mathematics Physics

**Mathematics**



**Physics**

# The Grand Canyon



# QUESTIONS

**I:** Suppose $T(x, y) = k(x^2 + y^2)$ where $k$ is a constant.
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What is $T(r, \theta)$?

## Choices

**A:** $T(r, \theta) = kr^2$

**B:** $T(r, \theta) = k(r^2 + \theta^2)$

**MATH**

$$T = f(x, y) = k(x^2 + y^2)$$
$$T = g(r, \theta) = kr^2$$

**PHYSICS**

$$T = T(x, y) = k(x^2 + y^2)$$
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**MATH**

$$T = f(x,y) = k(x^2+y^2)$$
$$T = g(r,\theta) = kr^2$$

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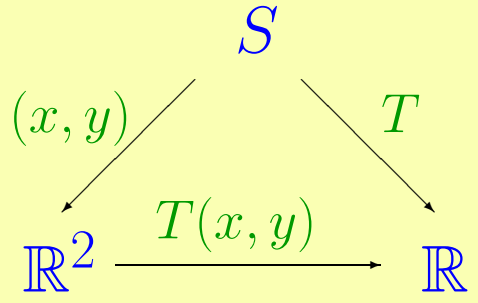
**Differential Geometry!**

$$T(x,y) \longleftrightarrow T \circ (x,y)^{-1}$$
$$T(r,\theta) \longleftrightarrow T \circ (r,\theta)^{-1}$$



## EXAMPLE

*(Stewart 4th edition §15.6: 12, 13, 14)*

Find the directional derivative of the function at the given point in the direction of $\vec{\boldsymbol{v}}$.

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(c) $g(r, \theta) = e^{-r} \sin\theta$, $(0, \pi/3)$, $\vec{\boldsymbol{v}} = 3\,\hat{\boldsymbol{\imath}} - 2\,\hat{\boldsymbol{\jmath}}$

## I: Physics is about things.

(a) What sort of a beast is it?

- Is it a scalar or a vector?
- What are the units? ($x$ is a length!)

(b) Physics is independent of coordinates.

- Vectors are arrows in space, not triples of numbers.
- Dot and cross products defined (and computed) geometrically.

(c) Graphs are about the relationships of physical things.

- Most physics is three-dimensional.
- Points in the domain of a function physically exist.
- The values of a function represent some measurable quantity.
- Hills are not good examples of functions of two variables.
- 3-d graphs of functions of 2 variables are misleading. (Use color!)

(d) Fundamental physics is highly symmetric.

- Spheres and cylinders vs. paraboloids.
- Interesting physics problems can involve trivial math.
- Use of $\hat{\boldsymbol{r}}$, $\hat{\boldsymbol{\theta}}$, $\hat{\boldsymbol{\phi}}$.

## II: Physicists can't change the problem.

(a) Physics involves the creative synthesis of multiple ideas.

- Upper division vs. lower division.

(b) Physics problems may not be well-defined math problems.

- There may be no obvious coordinates or independent variables.
- A physical object rarely comes with a parameterization.
- The unknowns don't have names.
- Getting to a well-defined math problem is part of the problem.
- If you can't add units, it wouldn't make a good physics problem.

(c) Physics problems don't fit templates.

- Template problems not very useful.
- A few key ideas are more easily remembered later.

(d) Physics involves the interplay of multiple representations.

- Dot product.

# GOALS

Mathematics

- Visualization
- Arbitrary surfaces
- 2-d primary
- Sophistication now
- Domain is important
- End of course flexible

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**Mathematics**

- Visualization
- Arbitrary surfaces
- 2-d primary
- Sophistication now
- Domain is important
- End of course flexible

**Physics**

- Visualization
- Simple surfaces
- 3-d primary
- Sophistication later
- Surface is important
- End of course fixed

**CUPM**

MAA Committee on the Undergraduate Program in Mathematics

Curriculum Guide

http://www.maa.org/cupm/cupm2004.pdf

**CRAFTY**

Subcommittee on Curriculum Renewal Across the First Two Years

Voices of the Partner Disciplines

http://www.maa.org/cupm/crafty

# CUPM: Math vs. Physics

State problems carefully, modify problems when necessary to make them tractable, articulate assumptions, appreciate the value of precise definitions, reason logically to conclusions, and interpret results intelligently.

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Move from a statement of the problem in words to a mathematical model, idealize the model or approximate when necessary to make it tractable, articulate assumptions, appreciate that words have precise technical meanings, reason logically to conclusions, and interpret results intelligently.

# CUPM: Math Goals

**Approach problem solving with a willingness to try multiple approaches, persist in the face of difficulties, assess the correctness of solutions, explore examples, pose questions, and devise and test conjectures.**

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**Yes! Yes! Yes!**

# CUPM: Physics Priority Topics

## Highest:

- Behavior of Simple Functions
- Derivatives and Integrals of Simple Functions

## Higher:

- Limiting Cases
- Differential Equations
- Max/Min Problems
- Line and Surface Integrals
- Vector Operations

## High:

- Series and Sequences
- Polar and other Coordinate Systems.

# THE BRIDGE PROJECT

- Small group activities
- Instructor's guide
- Study guide
- CWU, MHC, OSU, UPS, UWEC
- Workshops
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- **Differentials** *(Use what you know!)*
- **Multiple representations**
- **Symmetry** *(adapted bases, coordinates)*
- **Geometry** *(vectors, div, grad, curl)*

# PUBLICATIONS


1. Tevian Dray and Corinne A. Manogue, *The Vector Calculus Gap: Mathematics ≠ Physics*, PRIMUS **9**, 21–28 (1999).

2. Tevian Dray and Corinne A. Manogue, *Electromagnetic Conic Sections*, Am. J. Phys. **70**, 1129–1135 (2002).

3. Tevian Dray & Corinne A. Manogue, *Spherical Coordinates*, College Math. J. **34**, 168–169 (2003).

4. Tevian Dray & Corinne A. Manogue, *The Murder Mystery Method for Determining Whether a Vector Field is Conservative*, College Math. J. **34**, 238–241 (2003).

5. Tevian Dray and Corinne A. Manogue, *Using Differentials to Bridge the Vector Calculus Gap*, College Math. J. **34**, 283–290 (2003).

# SMALL GROUP ACTIVITIES

**Task Master:** *The task master ensures that the group completes all of the parts of the work.*

"Part 1 says that we must ... . How shall we do it?"

"What you had for lunch doesn't seem relevant. Can we get back to the main question?"

**Cynic:** *The cynic questions everything the group does and ensures that everyone in the group understands what is going on.*

"Why are we doing it this way?"

"Wouldn't it be better if we did it this other way?"

"I don't understand this part of what we are doing, let's go over it again."

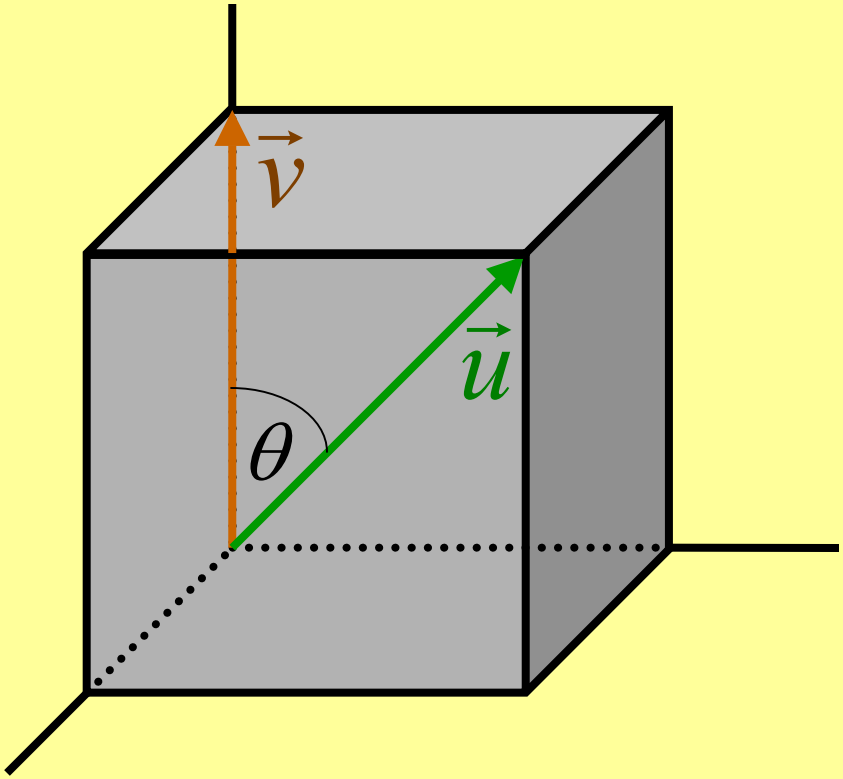
**Recorder:** *The recorder records the group's answers.*

"Do we agree that the answer to Part 3 is ... ?"

"I have written ... for Part 2. Is that what we want to say?"

**Reporter:** *The reporter reports to the class.*

# The Cube




$$\vec{u} = \hat{\imath} + \hat{\jmath} + \hat{k}$$

$$\vec{v} = \hat{k}$$

$$\cos(\theta) = \frac{\vec{u} \cdot \vec{v}}{\left|\vec{u}\right|\left|\vec{v}\right|} = \frac{1}{\sqrt{3}}$$

**Find the angle between a diagonal of a cube and an edge**

# The Cube

- **Emphasizes that vectors are arrows**
- **Combines geometry and algebra**
- **Uses multiple representations**

geometry: $\vec{u} \cdot \vec{v} = \left|\vec{u}\right|\left|\vec{v}\right| \cos\theta$

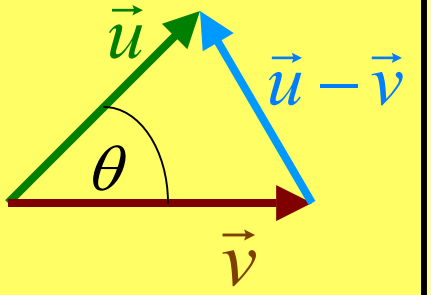
algebra: $\vec{u} \cdot \vec{v} = \left(u_1 \hat{\imath} + u_2 \hat{\jmath} + u_3 \hat{k}\right) \cdot \left(v_1 \hat{\imath} + v_2 \hat{\jmath} + v_3 \hat{k}\right)$

memory: $\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2 + u_3 v_3$

# Start with Algebra?

$$\vec{u}\cdot\vec{v} = u_1v_1 + u_2v_2 + u_3v_3$$

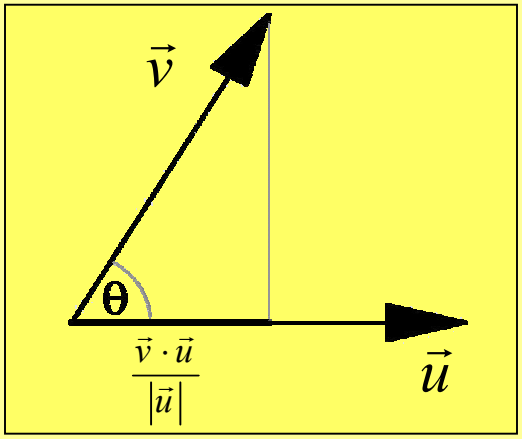
$$\Rightarrow (\vec{u}-\vec{v})\cdot(\vec{u}-\vec{v}) = \vec{u}\cdot\vec{u} + \vec{v}\cdot\vec{v} - 2\vec{u}\cdot\vec{v}$$

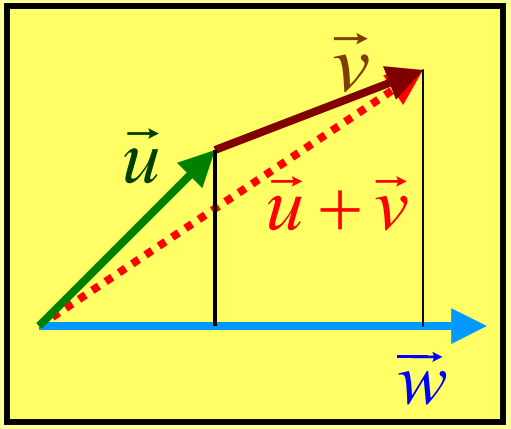



**(Law of Cosines)**

$$\left|\vec{u}-\vec{v}\right|^2 = \left|\vec{u}\right|^2 + \left|\vec{v}\right|^2 - 2\left|\vec{u}\right|\left|\vec{v}\right|\cos\theta$$

# Start with Geometry!



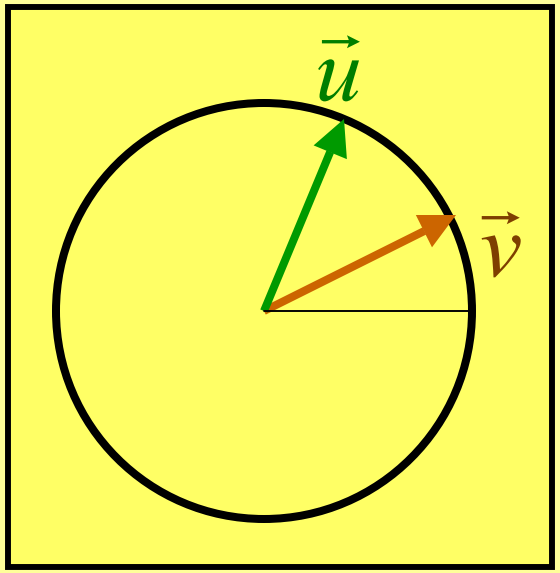




$$\Rightarrow (\vec{u}+\vec{v})\cdot\vec{w} = \vec{u}\cdot\vec{w} + \vec{v}\cdot\vec{w}$$

$$\Rightarrow \left(u_1\hat{\imath}+u_2\hat{\jmath}\right)\cdot\left(v_1\hat{\imath}+v_2\hat{\jmath}\right) = u_1v_1 + u_2v_2$$

**(get Law of Cosines for free!)**

# Use both!




$$\vec{u} = \cos(\alpha)\,\hat{i} + \sin(\alpha)\,\hat{j}$$

$$\vec{v} = \cos(\beta)\,\hat{i} + \sin(\beta)\,\hat{j}$$

$$\vec{u} \cdot \vec{v} = \cos(\alpha - \beta)$$

**(get addition formulas for free!)**

# CORIANDER

**I:** In the small town of Coriander, the library can be found by starting at the center of the town square, walking 25 meters north, turning 90° to the right, and walking a further 60 meters.

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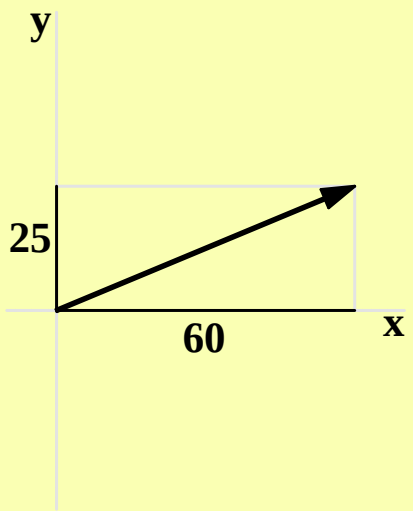


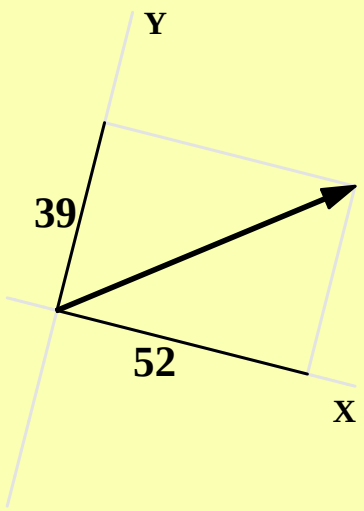

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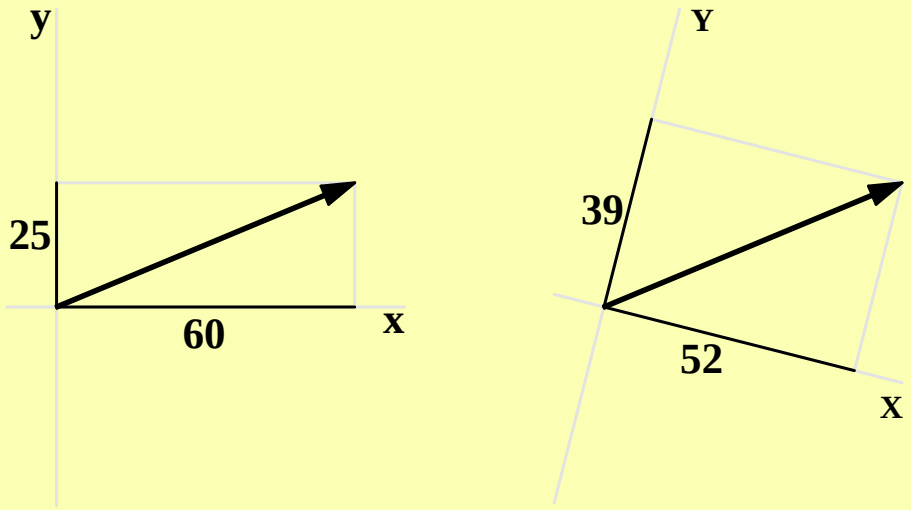
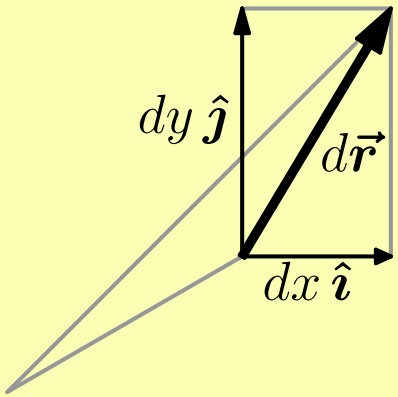
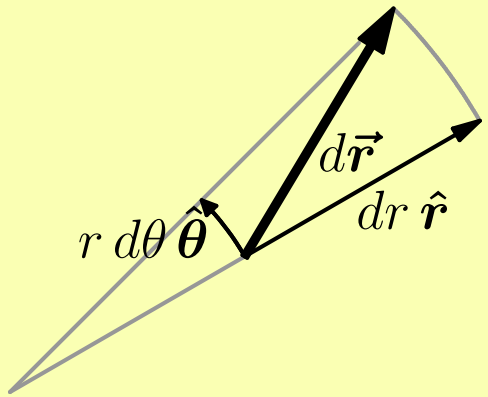



Where on Earth is Coriander?

# VECTOR DIFFERENTIALS




$$d\vec{\boldsymbol{r}} = dx\,\hat{\boldsymbol{\imath}} + dy\,\hat{\boldsymbol{\jmath}}$$




$$d\vec{\boldsymbol{r}} = dr\,\hat{\mathbf{r}} + r\,d\theta\,\hat{\boldsymbol{\theta}}$$

$$\begin{aligned}
ds &= |d\vec{\boldsymbol{r}}| \\
d\vec{\boldsymbol{S}} &= d\vec{\boldsymbol{r}}_1 \times d\vec{\boldsymbol{r}}_2 \\
dS &= |d\vec{\boldsymbol{r}}_1 \times d\vec{\boldsymbol{r}}_2| \\
dV &= (d\vec{\boldsymbol{r}}_1 \times d\vec{\boldsymbol{r}}_2) \cdot d\vec{\boldsymbol{r}}_3 \\
df &= \vec{\nabla} f \cdot d\vec{\boldsymbol{r}}
\end{aligned}$$

# A RADICAL VIEW OF CALCULUS

- The central idea in calculus is not the limit.
- The central idea of derivatives is not slope.
- The central idea of integrals is not area.
- The central idea of curves and surfaces is not parameterization.
- The central representation of a function is not its graph.

# A RADICAL VIEW OF CALCULUS

- The central idea in calculus is the differential.
- The central idea of derivatives is rate of change.
- The central idea of integrals is total amount.
- The central idea of curves and surfaces is "use what you know".
- The central representation of a function is data attached to the domain.

# SIMPLE CURRICULUM ADDITIONS

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- Integrals involve chopping things up and adding up a physical quantity on each piece.
- Limits (and approximations) require one to say what dimensionless quantity is small.

# MULTIPLE REPRESENTATIONS

Flux is the total amount of electric field through a given area.

Perpendicular Component

Normal Vector

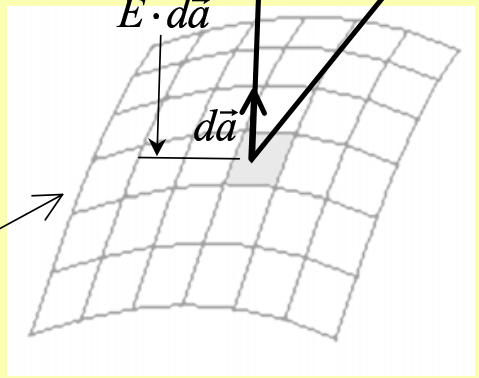
$$\Phi = \int \vec{E} \cdot d\vec{a}$$

$\vec{E} \cdot d\vec{a}$

$\vec{E}$

$d\vec{a}$

$\sum$ over all rectangles



# WORKSHOPS

- **Workshop (AAPT), Miami, FL, 1/24/04**
- **Workshop (PREP), South Hadley, MA, 6/18–22/04**
- **Minicourse (PNWMAA), Anchorage, AK, 6/24/04**
- **Faculty Workshop (NSF), Corvallis, OR, 8/7–11/04**
- **Minicourse (MathFest), Providence, RI, 8/12–14/04**
- **Minicourse (Joint Meetings), Atlanta, GA, 1/5–8/05**

(funding for local expenses available)