THE IMPORTANCE OF GEOMETRIC REASONING

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I: Mathematics $\neq$ Physics
II: The Bridge Project
III: Geometric Visualization

Why physics?

If we can't talk to physicists, who can we talk to?!

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Choices

A: $T(r, \theta) = kr^2$

B: $T(r, \theta) = k(r^2 + \theta^2)$

EXAMPLE

(Stewart 4th edition §15.6: 12, 13, 14)

Find the directional derivative of the function at the given point in the direction of $\vec{\boldsymbol{v}}$.

(a) $f(x, y) = x/y$, $(6, -2)$, $\vec{\boldsymbol{v}} = \langle -1, 3 \rangle$

(b) $g(s, t) = s^2 e^t$, $(2, 0)$, $\vec{\boldsymbol{v}} = \hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}}$

(c) $g(r, \theta) = e^{-r} \sin\theta$, $(0, \pi/3)$, $\vec{\boldsymbol{v}} = 3\,\hat{\boldsymbol{\imath}} - 2\,\hat{\boldsymbol{\jmath}}$

MATH

$$T = f(x,y) = k(x^2 + y^2)$$
$$T = g(r,\theta) = kr^2$$

PHYSICS

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MATH

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\begin{aligned}
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Differential Geometry!

$$
\begin{aligned}
T(x,y) &\longleftrightarrow T \circ (x,y)^{-1} \\
T(r,\theta) &\longleftrightarrow T \circ (r,\theta)^{-1}
\end{aligned}
$$

$$
\begin{tikzcd}
 & S \arrow[dl, "{(x,y)}"'] \arrow[dr, "T"] & \\
\mathbb{R}^2 \arrow[rr, "{T(x,y)}"] & & \mathbb{R}
\end{tikzcd}
$$

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III: What is $\vec{\nabla} f$ if $f(x) = x$?

I: Physics is about things.

(a) What sort of a beast is it?

- Scalar fields
- Coordinates
- Units
- Time

(b) Physics is independent of coordinates.

- Vectors as arrows
- Geometry of dot and cross products

(c) Graphs are about the relationships of physical things.

- Fundamental physics is three dimensional.
- 3d graphs of functions of 2 vars are misleading
- Hills are not good examples of functions of two variables
- Use of color
- Graphs of waves are misinterpreted

(d) Fundamental physics is highly symmetric.

- Spheres and cylinders vs. paraboloids
- Interesting physics problems can involve trivial math
- Use of $\hat{\boldsymbol{r}}$, $\hat{\boldsymbol{\theta}}$, $\hat{\boldsymbol{\phi}}$

II: Physicists can't change the problem.

(a) Physics involves the creative synthesis of multiple ideas.

(b) Physics problems may not be well-defined math problems.

- No preferred coordinates or independent variables.
- No parameterization.
- Unknowns don't have names.
- Getting to a well-defined math problem is part of the problem
- If you can't add units, it's a poor physics problem.

(c) Physics problems don't fit templates.

- Template problem-solving vs. skills
- A few key ideas are remembered best later

(d) Physics involves the interplay of multiple representations.

- Dot product

THE BRIDGE PROJECT

- NSF/DUE 0088901
- OCEPT
- Small group activities
- Instructor's guide (in preparation)
- CWU, MHC, OSU, UPS, UWEC
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Why vector calculus?

- Transition between lower- and upper-division coursework
- Transition between mathematics and other disciplines

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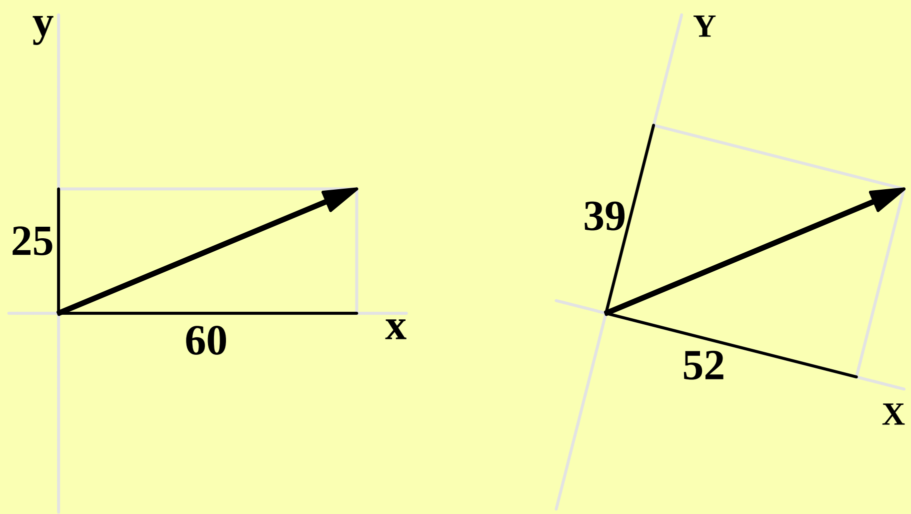

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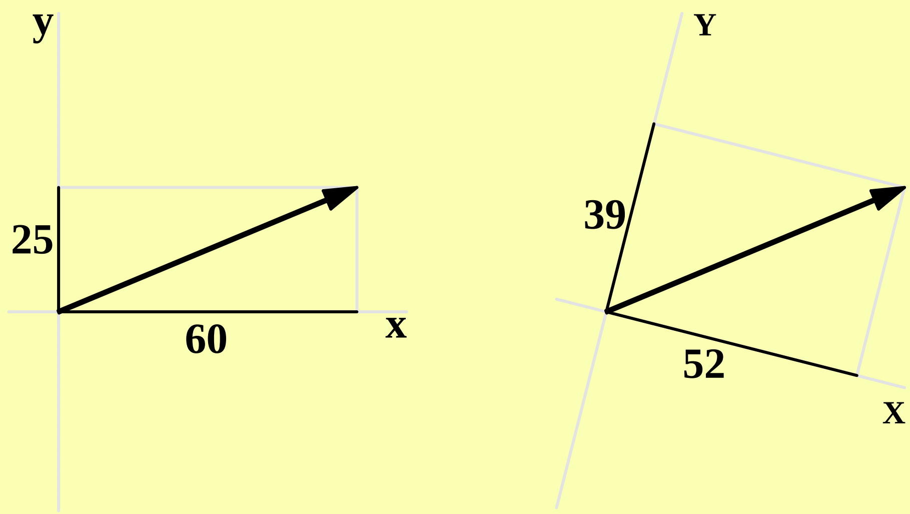
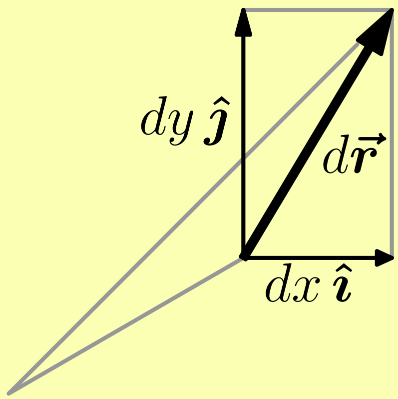
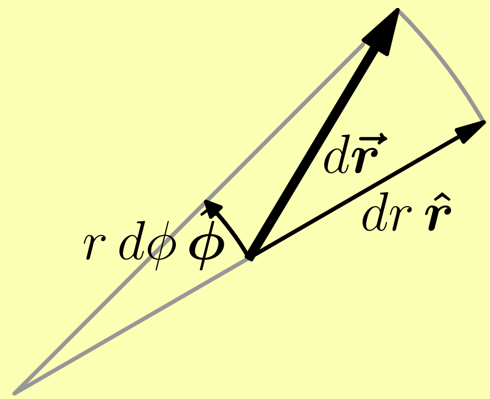


Where on Earth is Coriander?

VECTOR DIFFERENTIALS



$$d\vec{\boldsymbol{r}} = dx\,\hat{\boldsymbol{\imath}} + dy\,\hat{\boldsymbol{\jmath}}$$



$$d\vec{\boldsymbol{r}} = dr\,\hat{\boldsymbol{r}} + r\,d\theta\,\hat{\boldsymbol{\theta}}$$

$$\begin{aligned}
ds &= |d\vec{\boldsymbol{r}}| \\
d\vec{\boldsymbol{S}} &= d\vec{\boldsymbol{r}}_1 \times d\vec{\boldsymbol{r}}_2 \\
dS &= |d\vec{\boldsymbol{r}}_1 \times d\vec{\boldsymbol{r}}_2| \\
dV &= (d\vec{\boldsymbol{r}}_1 \times d\vec{\boldsymbol{r}}_2) \cdot d\vec{\boldsymbol{r}}_3 \\
df &= \vec{\nabla} f \cdot d\vec{\boldsymbol{r}}
\end{aligned}$$

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- Hard to teach, hard to write down — but worth it!