Integrals in Mathematics and Physics

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Abstract

We argue that the common thread connecting integration as taught in introductory calculus with integration as used in other disciplines, notably physics, is *chop, multiply, add*. We compare this unifying paradigm with other frameworks used to describe single-variable integration, and also address the generalization to multiple integration.

1 Introduction

Many attempts have been made to analyze student reasoning around single-variable integration, leading also to several quite different instructional strategies. There are, essentially, two major approaches (Jones and Ely, 2023), namely *accumulation from rate* (AR), introduced by Thompson (1994), and *adding up pieces* (AUP), as described by Jones (2013, 2015). A more detailed comparison of these two approaches was given by Jones and Ely (2023).

In AR, a (single) integral is viewed as an *accumulation*. What is being accumulated? The Fundamental Theorem of Calculus (FTC) tells us the answer: the integrand is the derivative of the integral. Thus, according to AR, a (single) integral is the accumulation of the derivative. The canonical example is taken from physics, with position being viewed as the accumulation of velocity.

In AUP, on the other hand, an integral is viewed as a *sum*. What is being summed? Not merely the integrand, but also the measure (e.g. dx). Thus, each piece being added is a *product*, which we choose to emphasize by describing the process of integration as *chop, multiply, add*.

We argue here that AR fails to capture features of integration that are essential in applications in other disciplines, notably including physics, from which the canonical example was taken. In a nutshell, there are two issues with AR. First, there is no FTC in higher dimensions, so AR fails to generalize to multiple integration. But even in one variable, not all integrands can be reasonably interpreted as physical rates. Although using the FTC in such contexts can not be faulted on mathematical grounds, it fails to support the expert reasoning used in other disciplines.

What about AUP? Neither of these issues arise. Yes, the pieces are now multidimensional, but the explicit product with the measure generalizes naturally to higher dimensions. And since the FTC was never emphasized, integrands were not interpreted as rates.

2 Mathematics vs. Physics

Science is about the world we observe, that is, the things we can *measure*. Physical quantities are related by equations that model how these quantities vary with respect to each other; *covariation* is indeed at the heart of science. However, which of these quantities are independent depends on the experiment being run; the independent variables are the adjustable inputs to the experiment. Sometimes, the measured quantity is itself a derivative. A radar speed gun measures the Doppler shift in the frequency of its laser beam, which depends directly on the speed. Other times, the derivative must be computed. If you use an app on your smartphone to "measure" your speed, you're actually measuring small changes in your position, and dividing the result by the corresponding small change in time. That is, you are *measuring* position, but *computing* speed. So whether you are actually measuring velocity directly depends on the experiment.

Position is thus an example of a physical quantity that *sometimes* can be thought of as accumulating from a rate, in this case the velocity, which is the derivative of position. But there are other physical quantities, related by integration, where the interpretation as an accumulation is at best contrived; it can be done in principle, but is not natural. In other words, it is not a perspective that experts are likely to recognize, let alone use.

We give several examples below.

2.1 Density

Another common example in physics is the relationship between the mass of a wire and its (linear) density. Yes, of course, the total mass of the wire is the integral of the density along the wire. But there is no physical sense in which this mass is "accumulating", apart from the purely theoretical statement based on the FTC that holds only in the single-variable case.

One can imagine chopping up the wire into small pieces, weighing each of them separately, and thus determining the density. The total mass could then be obtained by multiplication (with the length of each piece) and addition. Mind you, it would have been easier to have weighed the wire without chopping it up! However, it is considerably more challenging to imagine determining the *change* in mass of the wire due to a small change in length, in an attempt to obtain the density as the derivative of some "mass function".

Put differently, there is no physical "mass function" that describes the object, whose derivative would be the density. Rather, it is the density itself that is the fundamental physical quantity. Even in one dimension, mass may be the integral of density, but it is not natural to think of density as the derivative of mass. And even if one chooses to do so, presumably using the FTC, there is no generalization to higher dimensions.

2.2 Force

2.3 Current

Many students in university physics courses have already seen the relationship

$$I = \frac{dQ}{dt} \tag{1}$$

relating electric current I to moving charges. One can integrate this equation to obtain an expression for the total amount of charge Q passing through a given point on a wire in terms of the current flowing through the wire, namely

$$Q = \int I\,dt. \tag{2}$$

But neither the charge nor the current is "accumulating". What does one measure here? The current.

2.4 Internal Energy

The fundamental thermodynamic identity is

$$dU = T\,dS - p\,dV \tag{3}$$

where U is the internal energy, T is the temperature, S is the entropy, p is the pressure, and V is the volume. This identity tells us that the pressure is, up to sign, the derivative of the internal energy with respect to volume with entropy held constant, that is,

$$p = -\left(\frac{\partial U}{\partial V}\right)_S. \tag{4}$$

Thus, at constant entropy, the internal energy is (again, up to sign) the integral of the pressure, that is

$$U = -\int p\,dV \qquad (S = \text{const}). \tag{5}$$

But no thermodynamicist thinks of the internal energy as "accumulating" as the volume decreases. In fact, the actual value of the internal energy has no physical meaning; only the change in the internal energy between states matters. What does one measure here? The pressure and volume; the energy itself can not be measured.

References

Jones, S. R. (2013). Understanding the integral: Students' symbolic forms. *Journal of Mathematical Behavior*, 32(2):122–141. http://doi.org/10.1016/j.jmathb.2012.12.004.

Jones, S. R. (2015). Areas, anti-derivatives, and adding up pieces: Definite integrals in pure mathematics and applied science contexts. *The Journal of Mathematical Behavior*, 38:9–28. https://doi.org/10.1016/j.jmathb.2015.01.001.

Jones, S. R. and Ely, R. (2023). Approaches to integration based on quantitative reasoning: Adding Up Pieces and Accumulation from Rate. *Int. J. Res. Undergrad. Math. Ed.*, 9(1):8–35. https://doi.org/10.1007/s40753-023-00214-2.

Thompson, P. W. (1994). Images of rate and operational understanding of the fundamental theorem of calculus. *Educational Studies in Mathematics*, 25(3):165–208. https://doi.org/10.1007/BF01273664.