Riding a Bicycle Uphill

Tevian Dray
Department of Mathematics, Oregon State University, Corvallis, OR 97331, USA
tevian@math.oregonstate.edu

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Abstract

A mathematician ponders how to ride up a steep road that is tilted both ways.

Overview

There is a street near where I live that is too steep to ride my bicycle up. So I zigzag. The reader may be imagining a path similar to the switchbacks on a mountain road, but the street is also tilted sideways, so the path will not be symmetric around the direction of the street. What can we say about the best way to ride up the street?

Algebra

As can be seen in Figure 1, the street has nearly constant slope, so it is essentially planar. The equation of a (non-vertical) plane is

$$z = ax + by + c \tag{1}$$

and we choose coordinates so that x runs horizontally across the street (to the right), y runs horizontally "up" the street, and z, of course, is vertical. There are two slopes here, namely the coefficients a and b, corresponding to the steepness "across" and "up" the street, respectively. [1]

These two slopes completely determine the orientation of the plane! You can demonstrate this fact by taking a planar object such as a thin notebook, holding it parallel to the floor, and tilting it first left-to-right, then (without reverting to the original position) front-to-back. Any desired orientation of the notebook can be achieved by suitable tilts.

Steepness could be described by the angles α and β implicitly defined by $a = \tan\alpha$, $b = \tan\beta$. However, in the context of roads, steepness is usually given directly in terms of the slopes themselves. The *grade* of a road is defined as its slope converted to a percentage.

[1] These coefficients are the *partial derivatives* of the height function z with respect to x (with y held constant) and y (with x held constant), respectively.

Figure 1: A steep street near where the author lives, tilted in both directions.

Thus, a 100% grade corresponds to a 45° angle—an impossibly steep hill for a road. The steepest grade on the mountain passes on the interstate highway near where I live is about 6%. There's chatter on the internet about folks riding up grades of more than 25%, but anything over 10% is *really* steep. For comparison, Google Maps claims "my" hill rises 33 feet in 443 feet, which works out to just under 7.5%. [2]

In order to determine the best path up the street, we need to know the slope in an arbitrary (horizontal) direction. Even though the street is planar, we will use infinitesimal reasoning, a strategy that is also useful in other contexts [1, 2, 3]. If x and y vary by small amounts dx and dy, respectively, then the small change dz in z is

$$dz = a\,dx + b\,dy. \tag{2}$$

A (horizontal) direction is given by relating dy to dx. If $dy = m\,dx$, then (2) reduces to

$$dz = (a + bm)\,dx. \tag{3}$$

To figure out the *slope* in this direction, though, we need to compare the rise (dz) to the run—which is not dx. Rather, the horizontal distance traveled is the *arclength* ds, defined by

$$ds^2 = dx^2 + dy^2. \tag{4}$$

Putting this all together, the slope in the direction determined by $dy = m\,dx$ is given by

$$M = \frac{a + bm}{\sqrt{1 + m^2}}. \tag{5}$$

Now, suppose that the steepest hill I can cycle up has slope M_0. We are assuming that $M_0 < b < M_{\text{max}}$, so that I have to zigzag. Which way should I go?

[2]For such a small angle, it doesn't matter whether the distance given by Google Maps is the actual distance traveled or the horizontal displacement.

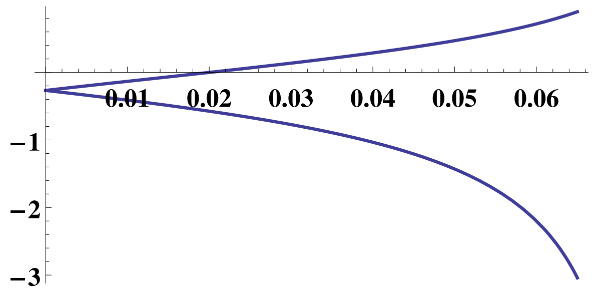

Figure 2: The two values of m (vertical axis) corresponding to the maximum slope M_0 (horizontal axis) that the author can ride up. The hill is described by (1) with $a = 0.02$ and $b = 0.075$. Riding on a path with $dy = m\,dx$ yields an actual slope of M_0 as in (6).

We can solve (5) with $M = M_0$ for the corresponding value of m, yielding

$$m = \frac{-ab \pm M_0\sqrt{a^2 + b^2 - M_0^2}}{b^2 - M_0^2} \tag{6}$$

For my hill, we have $b = 0.075$. The grade across the hill is perhaps 2%, so $a = 0.02$. We can plot m as a function of M_0, as shown in Figure 2. Assuming that the steepest hill that I can ride up has a grade of 4%, it's clear that I should go nearly straight across the street (m small and positive; $\arctan(m) \approx 16°$) when riding to the right, but diagonally across the street ($m \approx -1$; $\arctan(|m|) \approx 46°$). This path is shown in Figure 3

The alert reader will have noticed that Figure 2 has a peculiar property: If $M_0 < a$, then *both* values of m are negative. What does this mean? Surely it's not possible to always zigzag to the left!

Further reflection resolves this apparent contradiction. The upper graph in Figure 2 always corresponds to going to the right; the sign of m then tells you whether you're going up or down the street. If $M_0 = a$, then you should ride exactly straight across the street to the right, but if $M_0 < a$, then you should actually backtrack "down" the street—although you'd still be going uphill. [3]

Geometry

What is the steepest slope, and in which direction is it? We can optimize M as a function of m. After a slightly annoying computation (due to the square root), we discover that M obtains its maximum value when $m = b/a$, so that

$$M_{\max} = \sqrt{a^2 + b^2} \tag{7}$$

and the steepest direction is given by $ay - bx = \text{const}$, or, equivalently, by the vector $\vec{\boldsymbol{v}} = a\,\hat{\boldsymbol{x}} + b\,\hat{\boldsymbol{y}}$.

[3]This actually happens to me on a regular basis, since M_0 is not really a constant. If I lose too much momentum when making a turn, my effective maximum slope decreases, which can easily push my effective M_0 below a.

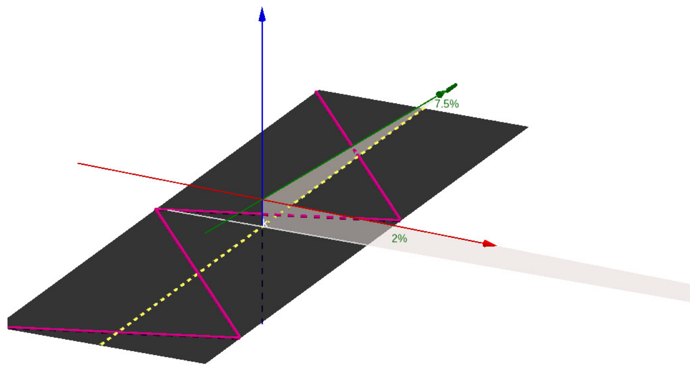

Figure 3: A model for the street, showing the tilt in both directions. The dashed yellow line goes "up" the street; the thin white line goes across the street, and the pink path represents the theoretical path the author should take. (In practice, the author rarely achieves this path.)

Is there a better way? Yes: Compute the *gradient* of the function z, yielding

$$\vec{\boldsymbol{\nabla}} z = \frac{\partial z}{\partial x}\,\hat{\boldsymbol{x}} + \frac{\partial z}{\partial y}\,\hat{\boldsymbol{y}} = a\,\hat{\boldsymbol{x}} + b\,\hat{\boldsymbol{y}}. \tag{8}$$

which already yields the *direction* of steepest slope; taking the magnitude $|\vec{\boldsymbol{\nabla}} z|$ yields the maximum slope.

It is a remarkable fact that the linearity of a plane implies that the slope in any direction depends only on the maximum slope and the angle from its direction. A small vector displacement in an arbitrary (horizontal) direction is given by

$$d\vec{\boldsymbol{r}} = dx\,\hat{\boldsymbol{x}} + dy\,\hat{\boldsymbol{y}} \tag{9}$$

whose magnitude is just the arclength ds given by (4). Comparing (8) with (2), we have

$$dz = \vec{\boldsymbol{\nabla}} z \cdot d\vec{\boldsymbol{r}}. \tag{10}$$

The slope of the hill in the given direction is the rise over the run, namely

$$\frac{dz}{ds} = \vec{\boldsymbol{\nabla}} z \cdot \frac{d\vec{\boldsymbol{r}}}{ds} = |\vec{\boldsymbol{\nabla}} z| \cos\theta \tag{11}$$

where θ is the angle between the given direction and the steepest direction, and where we have used the fact that $ds = |d\vec{\boldsymbol{r}}|$. Thus, the angle θ_0 corresponding to a given slope M_0 satisfies

$$\cos\theta_0 = \frac{M_0}{M_{\max}} = \frac{M_0}{\sqrt{a^2 + b^2}}. \tag{12}$$

Using the data above, we have $a = 0.02$, $b = 0.075$, $M_0 = 0.04$, resulting in $\theta_0 \approx 59°$. Since the steepest direction is given as a vector by (8), the corresponding angle $\theta_{\max}$ is given by

$$\tan\theta_{\max} = \frac{b}{a}, \tag{13}$$

so $\theta_{\max} \approx 75°$. Adding and subtracting these angles yields the same answer as before. In both cases, these angles are *horizontal*, that is, measured in the xy-plane from the positive x-axis. The geometric approach clearly requires less computation to determine these angles than the algebraic approach.

It's straightforward but somewhat messy to show that the geometric approach always agrees with the algebraic one. From (12) we have

$$\tan\theta_0 = \frac{\sqrt{a^2 + b^2 - M_0^2}}{M_0}. \tag{14}$$

Thus, the two directions corresponding to slope M_0 satisfy

$$\begin{aligned} m = \tan(\theta_{\max} \mp \theta_0) &= \frac{\tan\theta_{\max} \mp \tan\theta_0}{1 \pm \tan\theta_{\max}\tan\theta_0} \\ &= \frac{bM_0 \mp a\sqrt{a^2 + b^2 - M_0^2}}{aM_0 \pm b\sqrt{a^2 + b^2 - M_0^2}} \end{aligned} \tag{15}$$

which, after rationalizing the denominator, agrees with (6).

Summary

As constructed here, the best path up the street is symmetric around the direction of steepest slope, deviating from that direction just enough to make the slope manageable. In practice, I rarely achieve the optimal path, although I manage it more often on the (longer) left-hand legs than on the (shorter) right-hand legs. Put differently, the answer is subjective; paraphrasing a comment by one of my students when confronted with a similar problem [4], it depends both on the shape of the hill and the shape of the rider.

Disclosures

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Biography

Tevian Dray studied mathematics at MIT and Berkeley, spent several years as a physics postdoc, and is now Professor Emeritus of Mathematics at Oregon State University. Both his traditional research in mathematical physics and his efforts to improve the teaching of second-year calculus put geometry front and center. He has written several books emphasazing geometry, both print and online.

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