PRIMUS, 29(8): 830–850, 2019

ISSN: 1051-1970 print / 1935-4053 online
DOI: 10.1080/10511970.2018.1532934

Interpreting Derivatives

Tevian Dray, Elizabeth Gire, Mary Bridget Kustusch, Corinne A. Manogue and David Roundy

Abstract: Calculus, as commonly taught, describes certain operations on explicit functions, but science relies on experimental data, which is inherently discrete. In the face of this disparity, how can we help students transition from lower-division mathematics courses to upper-division coursework in other STEM disciplines? We discuss here our efforts to address this issue for upper-division physics majors by introducing a new representation for derivatives in terms of experiments to go along with the traditional symbolic, graphical, verbal, and numerical representations, and by emphasizing infinitesimal reasoning through the use of differentials. These ideas culminate in the concept of *thick derivatives*. By providing examples of "physics" reasoning about both ordinary and partial derivatives, and methods for incorporating such reasoning into the classroom, we hope to give instructors of calculus new insight into the needs of many of their students.

Keywords: Derivatives, differentials, experiment, measurement, physics

1. INTRODUCTION

We are an interdisciplinary team of mathematics and physics educators, with backgrounds in mathematics, theoretical and computational physics, and physics education. We have worked together for many years designing curricula at the "middle-division" level, consisting of second-year calculus and third-year physics courses [3, 33, 50]. We are members

Address all correspondence to Tevian Dray, Department of Mathematics, Oregon State University, Corvallis, OR 97331, USA. E-mail: tevian@math.oregonstate.edu
Color versions of one or more of the figures in the article can be found online at www.tandfonline.com/upri.

of the team that redesigned all upper-division physics courses at Oregon State University, a group that has met monthly since 1996 to discuss course design and implementation, including pedadogy, in great detail.

Many of our group interactions have focused on trying to understand how each of us uses mathematics to solve typical problems in our various subdisciplines. We have been shocked by the differences; our initial reactions can only be described as stunned disbelief that somebody else would actually think that way. However, the strength of our group dynamics, developed over many years, leads us quickly into robust conversations about the value each (sub)discipline finds in the concepts, language, and notation it employs. The group is anchored by a husband and wife team (TD & CAM), who are highly motivated to move the discussion beyond the possible rancor which can be associated with the initial disbelief. These discussions in turn have led to a recognition that such disciplinary differences represent a significant challenge for students, who may need to rapidly assimilate the perspectives of multiple disciplines. In this paper, we share some of the surprising things we have learned, together with some pedagogical strategies intended to help students bridge these disciplinary gaps.

Paraphrasing Winston Churchill, we have found it useful to describe mathematics and physics as two disciplines separated by a common language. Calculus is about functions, but the language of functions is not always a good match for the description of physical quantities. Physics describes the real world by finding relationships between these quantities. Theoretical descriptions must always be compatible with observation, that is, with experimental data. The smooth functions used as the starting point in calculus are instead, for most other scientists and engineers, the result of an idealization process that started with observation. Idealization is a powerful tool, but one should not lose sight of where the process began.

For many years, we have attempted to address this disparity in our own teaching by emphasizing geometric reasoning, in addition to symbolic manipulation [10, 13, 15]. We have argued [10] that mathematicians tend to emphasize the symbolic manipulation of *functions*, whereas physicists tend to care more about the *equations* relating physical quantities. We introduced a simple example that illustrates this difference. Temperature may be expressed by different functions in rectangular and polar coordinates (e.g., $T(x, y) = k(x^2 + y^2)$ and $T(r, \theta) = kr^2$), but these functions represent the same physical quantity, opening the door to a dispute [8, 12, 25, 34] over whether both of these functions can be called "T." Mathematicians and physicists may share a common vocabulary, but they do not use the same grammar.

Such minor disputes over language reflect a deeper difference in viewpoint which can be characterized by saying that *Physicists primarily*

Figure 1. One of the transparent plastic surface models developed by Aaron Wangberg at Winona State University as part of the Surfaces project. Each of the six color-coded surfaces is dry-erasable, as are the matching contour maps, one of which is visible underneath the surface. For further details, see [49, 50].

do geometry, but *Mathematicians tend to teach algebra*. Geometry is the study of invariant objects, whose algebraic description might involve a choice of coordinates; both geometers and physicists have a notion of temperature T defined at each point that is more fundamental than any coordinate-based description.

Such reasoning has led us to emphasize differentials in both our calculus and physics courses, rather than functions [6, 15]. This approach appears to work very well in multivariable and vector calculus, with students seeing differentiation and integration for the second time (although we have not yet had similar success with beginning students). As part of our geometric approach to multivariable calculus, we also utilize the plastic, writable surfaces developed by Aaron Wangberg [49, 51] and shown in Figure 1, which engage students directly with both geometric and numerical representations of derivatives.

More recently, as we watched students struggle to apply these geometric ideas to upper-division physics classes such as thermodynamics, we were struck by the mismatch between the smooth functions analyzed in calculus and the experimental data collected in physics. As a result, we have argued [35] that the experimental context is not the same as the "numerical representation" as most mathematicians understand that term, such as in the "rule-of-four" [19].

We present in this paper a view of derivatives, both ordinary and partial, that emphasizes not only geometric reasoning skills, but also the experimental understanding used by physicists. We have come to see infinitesimal reasoning as the final step in a sequence of idealizations. Science begins with *experiment*, which involves *measurement*. To understand calculus in terms of measurements, one needs a notion of *good approximation* that can encompass "experimental" differentiation. We call this notion a *thick derivative*, because it is an approximation, not exact. In such circumstances, the tangent line is blurred by the experimental uncertainty. *Infinitesimal reasoning* is then an idealized process for manipulating these thick derivatives symbolically.

Although our own experience lies at the transition from mathematics to physics, we believe the skills we are advocating are also essential beyond physics, and in particular for most scientists and engineers. It is our hope that the discussion below might provide instructors of calculus with new insight into the needs of many of their students.

2. THE CONTRAST BETWEEN MATHEMATICS AND PHYSICS

Physics is about things [11]. Physics *always* has a context; the quantities being studied refer to real attributes of real objects. Symbols mean something: x is typically a length; t a time. Physicists are, by necessity, bilingual, but "$\sin(x)$" is pure mathematics, since you cannot take sine of a dimensionful quantity. Units matter; a pendulum will be described by terms such as $A\sin(\omega t)$, where ω has the dimensions of inverse time. Such parameters are ubiquitous in science — and missing from the problems in many math texts.

We ask our students, "What sort of a beast is it?" referring to the nature of the physical quantities being represented by algebraic symbols. Not only does this question help students get the units right, a technique known as *dimensional analysis*, it also catches obvious, but common, errors such as setting vectors equal to scalars, or comparing finite quantities with infinitesimals.

Furthermore, scientists must deal with the world as it is. Even though physicists usually do assume that the underlying physical relationships are smooth, most smooth functions can not be written in terms of known functions. Scientific knowledge is obtained or verified using experimental data, which usually consists of discrete data points. Furthermore, quantum mechanics tells us that there is no sense in which one can actually take a limit to zero. Physicists and other scientists must therefore become experts at knowing *in a given context* which assumptions are reasonable, and which are not.

These differences in perspective between mathematicians and other scientists have significant implications for the teaching of mathematics at all levels. The most obvious suggestion is to include units — and dimensionful parameters — as a routine part of examples and problems. Another is to downplay the use of subtle counterexamples. Scientists never encounter nowhere-differentiable functions, nor do the coordinate singularities at the origin of polar coordinates cause much harm. Are the functions $x+2$ and $(x^2-4)/(x-2)$ really different? In a physical context where such differences matter, something will always signal this fact, such as the presence of a point charge or an infinite potential; otherwise, they can be safely ignored. The context of the real world provides an existence proof that eliminates most of the concerns about counterexamples. There is a fine line between teaching students to be careful, and teaching them to be scared of missing some mathematical detail that they are unlikely to actually encounter.

The calculus reform movement of the 1980s emphasized the importance of multiple representations [19, 46], but many mathematicians are still more inclined to emphasize symbolic and, perhaps, graphical representations than numerical representations, although the latter are very common in applications. Numerical representations rely on approximations, suggesting a need to expand the notion of derivative. We explore these ideas in the next two sections.

3. WHAT IS A DERIVATIVE?

Ask calculus students what a derivative is, and a common response will be “slope” [1, 5, 16, 32]. Yes, the slope of a graph represents a derivative. But what if there is no graph?

Previous work on student understanding of derivatives often uses the relatively narrow context of kinematics (motion) [2, 28]. Perceived student difficulties in transferring an elementary understanding of derivatives to applications has led to more recent work studying student understanding of derivatives in other contexts [4, 20, 30, 41].

Consider the apparatus shown in Figure 2, the *Partial Derivative Machine*, developed as a mechanical analog to problems in thermodynamics and intended to help students learn to reason about *partial* derivatives in a context where the functional dependencies between variables is not obvious. By pinning down one of the strings, we obtain the *Ordinary Derivative Machine*, consisting of a weight on a string connected to a nonlinear spring system, thus determining a relationship between the position x of the string (the location of the flag) and the tension F_x in the string (the attached weight). Given the task of determining the derivative of this position with respect to the tension, what can possibly be meant by such a “derivative?”

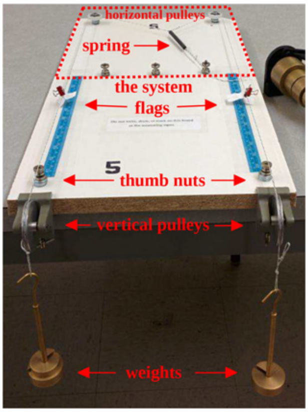


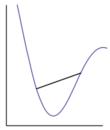
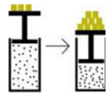
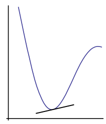
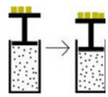
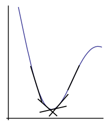
Figure 2. The Partial Derivative Machine designed by David Roundy at Oregon State University. In this mechanical analog of a thermodynamic system, the variables are the two string positions (the flags) and the tensions in the strings (the weights). However, it is not obvious how many independent variables there are, and which variables are considered independent depends on the context. For further details, see [37].

In a recent study [37], we asked faculty in mathematics, physics, and engineering to determine an analogous (partial) derivative using the Partial Derivative Machine. The only viable method for determining such derivatives is to measure both quantities while perturbing one of them — and holding an appropriate subset of the other variables fixed. The physicists and engineers were clearly familiar with this methodology, and had robust techniques for ensuring that their approximations were reasonable. The mathematicians, however, had difficulty engaging with the idea of a derivative that could not be obtained by an exact limit process.

So what is a derivative? A ratio of small changes in quantities? A ratio of *very* small changes in quantities? The slope of the tangent line is the limit of the slopes of secant lines. How does one take the limit of discrete, numerical data, such as that measured during an experiment?

Most mathematicians have a "bright line" test when it comes to derivatives. An *average* rate of change, no matter how small the domain, is different from an *instantaneous* rate of change. This distinction works fine for smooth functions, or graphs, but not very well for numerical data.

We believe that this bright line is in the wrong place. The most useful distinction is not whether a rate of change is average or instantaneous, but how good the approximation is. The quality of the approximation depends on the context; the constraints of the physical problem being

Process-object layer	Graphical	Verbal	Symbolic	Numerical	Physical
	Slope	Rate of Change	Difference Quotient	Ratio of Changes	Measurement
Ratio		"average rate of change"	$\frac{f(x+\Delta x)-f(x)}{\Delta x}$	$\frac{y_2 - y_1}{x_2 - x_1}$ numerically	
Limit		"instantaneous ..."	$\lim\limits_{\Delta x \to 0} \cdots$	...with Δx small	
Function		"... at any point/time"	$f'(x) = \ldots$	... depends on x	tedious repetition

	Symbolic
	Instrumental Understanding
Function	*rules to "take a derivative"*

Figure 3. An extended framework for the concept of the derivative [35].

solved usually tell the scientist what accuracy is needed. For example, if doubling or halving the step size (the denominator in the rate of change) consistently leads to the same ratio (within acceptable tolerances), it is reasonable to conclude that one is in the linear regime and has therefore obtained a good approximation to the derivative. We observed this strategy in engineers and physicists, but not mathematicians [37].

We are not suggesting that introductory calculus students become experts at numerical analysis, but instead that they be made aware that such approximations are a fundamental part of doing science. Rather than emphasizing the difference between instantaneous and average rates of change, students might be served better by emphasizing the need to arrive at answers that are "good enough," making clear that this notion depends on the context. For example, the upper left graph in Figure 3 might be a good approximation to the derivative at the point shown in the graph immediately below, but a terrible approximation to the derivative at either endpoint of the secant line.

Calculus need not be about formal limits, but rather the art of infinitesimal reasoning, where all that really matters about infinitesimals is that they are quantities that are "small enough" for the purpose at hand. To impose a sharp distinction between "average" and "instantaneous" makes it difficult to use the numerical and physical representations to find derivatives, as discussed in the next section.

4. THICK DERIVATIVES

Student understanding of the fundamental aspects of derivatives has been studied by several authors, including [18, 31, 52]. In particular, Zandieh [52] proposed a framework for student understanding of derivatives in single-variable calculus. Zandieh's framework draws on the idea of a *concept image* [48] (see also [42]), the set of properties associated with a concept together with mental pictures of the concept. We recently extended this framework [35], partly in an effort to include partial derivatives, but mostly to introduce a new *numerical* representation appropriate for data, in terms of the ratio of small changes. Our extended framework is shown in Figure 3. Each of the rows in Figure 3 describes a different *process-object layer* in the sense of Sfard's description [38] of the reification of processes into objects. The first (ratio) layer describes making a "(good) approximation"; the second (limit) layer describes taking a limit "at a point"; the third (function) layer describes extending the derivative to "many points." For further discussion, see [35].

Each of the columns in Figure 3 describes a different representation; we briefly discuss each representation in turn. The *graphical* representation of the derivative is slope, starting with the slope of a secant line, whose limit is the slope of the tangent line, and then extending this construction to a continuous domain of values of the independent variable. The *verbal* representation for the derivative is "rate of change," starting with average rate of change, then instantaneous rate of change, then recognizing that these rates of change exist at every point in the domain of the function. The *symbolic* representation of the derivative is its formal definition as the limit of a difference quotient, with the last step being the reinterpretation of the parameter (x) as an independent variable, rather than having a single value. The *numerical* representation of the derivative is a ratio of changes, computed from numerical data, with the limit process being reinterpreted as the changes being sufficiently small, and the final step again involving the reinterpretation of the parameter as a variable. Finally, we modified Zandieh's *physical* category to represent the mental process of designing and conducting an experiment which would result in the desired derivative [36, 40]. The limit process now corresponds to measurements made for nearly identical values of the parameter, and the reinterpretation as a function now requires "tedious repetition" to perform the necessary measurements. We follow Zandieh [52] and Likwambe and Christiansen [23] in treating separately the symbolic manipulations used to calculate derivatives, and include them in a separate category labeled *instrumental understanding*. Both Skemp [39] and Lithner [24] point out that instrumental understanding is commonly emphasized in both homework assignments and exams.

We emphasize that, for physicists, the distinction in Figure 3 between the Ratio and Limit process-object layers is not quite the bright line test between average and instantaneous rates of change used by most mathematicians, even though those words appear in the table. In order to include numerical and physical representations in the Limit layer, this bright line must be replaced by some notion of "small enough." We have therefore expanded the concept of derivative so as to encompass both the mathematicians' intent when taking limits and the scientist's need to work with discrete data. We refer to this expanded concept as *thick derivatives* [7, 35].

5. DIFFERENTIALS

A typical problem in thermodynamics is to determine some partial derivative, say $(\frac{\partial M}{\partial B})_S$, from given equations of state that, in this case, express the magnetization M of some material and the temperature T in terms of the magnetic field B and entropy S.

This notation for the partial derivative will be unfamiliar to many mathematicians and most students, with the subscript S indicating a partial derivative *with* S *held fixed*. Despite its unfamiliarity, this notation serves a crucial purpose. A partial derivative operator such as $\frac{\partial}{\partial x}$ is meaningless unless one knows what other variables are being held fixed. This essential feature of partial differentiation is often under-emphasized in multivariable calculus courses, where students may come away with the mistaken impression that a partial derivative with respect to x means that one should "hold everything else fixed." It may not be possible to do so! There are four related quantities in our example, M, B, T, and S, and it is not obvious how many of them are independent, much less which ones. In thermodynamics, *any* two of these four quantities could be treated as the independent variables, but it is not physically possible to vary three of them independently. (The Partial Derivative Machine discussed in Section 3 is similar; we deliberately do not specify which parameters are independent.)

When we gave a similar problem to several experts during interviews [21, 22], no two of them approached it the same way. We found three basic strategies, or "epistemic games," namely the use of substitution to isolate the independent variables, the use of the many partial derivative chain rules to express the desired derivative in terms of others, and the use of differentials to reduce the problem to one in linear algebra.

Given a consistent system of equations, one often attempts to use some of them (the "constraints") to eliminate "extra" dependent variables, resulting in a single remaining equation expressing some physical quantity in terms of one or more independent variables. In practice, however, the constraint equations may not be solvable, and, even if they are, the solutions may be unwieldy. Working with differentials, by contrast, leads to a set of linear equations that can always be solved — assuming an appropriate number of constraints.

As we discussed in [15], there are (at least) two quite different ways of interpreting differentials.[1] The most common approach is to decide first which variables are independent, and which are dependent. This use of *differentials of functions* is equivalent to *implicit differentiation*, in which, for example, the slope of a circle is found by differentiating both sides of

$$x^2 + y^2 = r^2, \tag{1}$$

with respect to the independent variable, usually x, yielding

$$2x + 2y\ \frac{dy}{dx} = 0. \tag{2}$$

We can, of course, rewrite equation (2) in terms of differentials, leading to

$$2x\ dx + 2y\ dy = 0, \tag{3}$$

which is completely symmetric in x and y. So why did we need to specify the independent variable(s)? We did not!

Start again, and "zap" both sides of equation (1) with d. Do not assume anything. The result is

$$2x\ dx + 2y\ dy = 2r\ dr, \tag{4}$$

since we have not (yet) assumed that the radius r is constant. Equation (4) therefore tells us how changes in radius are related to changes in x and/or y. We can recover the slope of the circle by further assuming r is constant, so that $dr = 0$, then solving for $\frac{dy}{dx}$. The use of *differentials of equations* postpones the discussion of dependent and independent variables until it is needed. This approach allows simultaneous consideration of the case where r is a function of x and y and the case where y is a function of x.

[1]Thompson and collaborators [43–45] have recently given a new interpretation of differentials which describes known smooth functions (e.g., $\sin(\theta)$) as piecewise linear on a "small enough" scale.

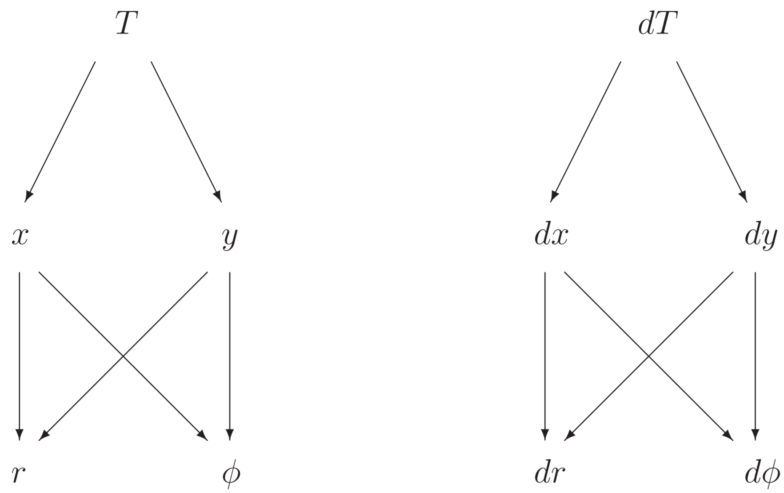


Figure 4. Shown on the left is the standard chain rule diagram for converting the temperature T in rectangular coordinates (x, y) to polar coordinates (r, ϕ). The diagram on the right shows the same situation, expressed in terms of differentials.

One great advantage of this technique is that the resulting equations, such as equation (4), always express linear relationships between differentials. Given a system of equations, their "zapped" versions can *always* be reduced using substitution.

One danger of this "use what you know" approach is that it is easy to lose track of what you know. Systematic substitution of differentials is always possible, but it is nonetheless easy (and common) to go in circles. A standard approach to help students through this maze is the use of *chain rule diagrams*, as shown in Figure 4, which serves as a shorthand reminder of the needed partial derivatives (see e.g., [29]). The information encoded in such diagrams underlies the second "game" we observed in experts, namely the use of (often memorized) chain rule identities. In the example described by Figure 4, we consider the temperature T on a plate in both rectangular and polar coordinates. A standard textbook calculation (see e.g., [29]) shows how to recover the chain rule expression

$$\frac{\partial T}{\partial r} = \frac{\partial T}{\partial x}\,\frac{\partial x}{\partial r} + \frac{\partial T}{\partial y}\,\frac{\partial y}{\partial r}, \tag{5}$$

using the left-hand diagram by thinking of the arrows as representing partial derivatives and following all possible paths from T to r, multiplying together the partial derivatives represented by each segment of a given path.

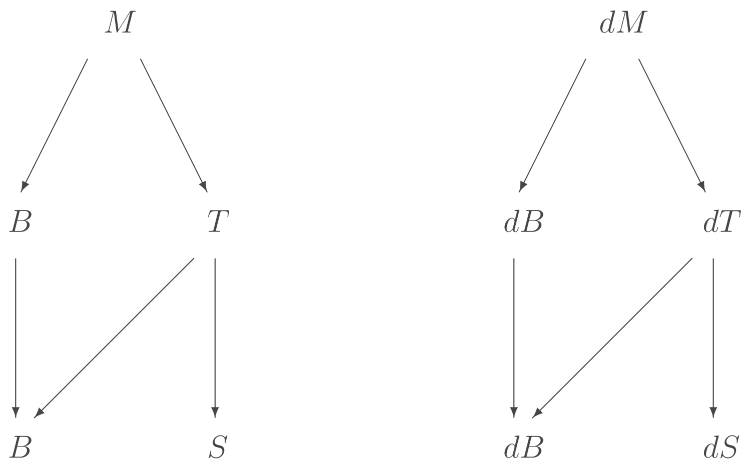


Figure 5. The chain rule diagram when changing only one of two variables, in this case expressing the magnetization M in terms of the magnetic field B and either the temperature T or the entropy S.

We prefer, however, to rewrite such diagrams in terms of differentials, in which case the diagram directly represents the linear relationships between the differentials of a given set of physical quantities. Thus, we replace the first diagram in Figure 4 by the second, which contains the same information. Using the right-hand diagram, we obtain equation (5) by comparing the coefficients in the linear relations

$$dT = \frac{\partial T}{\partial x}dx + \frac{\partial T}{\partial y}dy = \frac{\partial T}{\partial r}dr + \frac{\partial T}{\partial \phi}d\phi, \tag{6}$$

together with similar expressions for dx and dy in terms of dr and $d\phi$. These latter expressions can be evaluated using the known relationship

$$x = r\cos\phi \Rightarrow dx = \cos\phi \ dr - r\sin\phi \ d\phi, \tag{7}$$

$$y = r\sin\phi \Rightarrow dy = \sin\phi \ dr + r\cos\phi \ d\phi, \tag{8}$$

between rectangular and polar coordinates.

A more complicated situation involving the *adiabatic magnetic susceptibility* $(\frac{\partial M}{\partial B})_S$ is shown in Figure 5, in which the functional dependencies are more subtle. In this example, which is worked out in more detail in the Appendix, M is given as a function of B and T, but the *adiabatic* susceptibility is the partial derivative of M with respect to B *at constant entropy*. Since the magnetic field B is in both sets of independent variables, it is necessary to specify explicitly which variables are being held constant; the corresponding chain rule is

$$\left(\frac{\partial M}{\partial B}\right)_S = \left(\frac{\partial M}{\partial B}\right)_T + \left(\frac{\partial M}{\partial T}\right)_B \left(\frac{\partial T}{\partial B}\right)_S, \qquad (9)$$

obtained as before either by associating arrows with partial derivatives (using the left-hand diagram), or by expanding the differentials dM and dT in terms of dB and dS (using the right-hand diagram). In practice, the unwieldy partial derivative expressions occurring in equation (9) can often be avoided when working with differentials, using instead the explicit result of zapping the given equations of state with d.

We have suggested previously [6, 10, 15] that the use of differentials can provide students with a more robust understanding of calculus than traditional symbolic techniques. Differentials allow one to worry about the *linear* relationships between small changes in related quantities, rather than the usually much more complicated relationships between the quantities themselves. Infinitesimal reasoning is thus the art of linear approximation. Thermodynamics is perhaps unique in its frequent use of overlapping sets of independent variables, but it is precisely this sort of problem for which infinitesimal reasoning skills, as typified by the use of differentials, are most useful.

6. WHAT NEXT?

Readers might naturally be interested in further details about the nature of our collaboration, and in particular how we managed to sustain it for so long and the lessons that we learned, in order to apply these lessons to their own circumstances. However, the answers to these questions are long and complicated; our group is doing a separate research project to document these intricate relationships. A summary is available on the Paradigms project website [47]; a longer version will be posted there when available, and will ultimately be published separately.

The short answer involves frank and open discourse among a rotating group of diverse collaborators. Central to our research framework is that when we are trying to design curricular materials for particular groups of students we develop questions that we believe will be challenging for them and ask each other to solve them. Over and over again we have been stunned by each other's solution methods and strategies in a way that leads us to understand that no two experts solve problems the same way, even within, but certainly across, disciplines. No wonder our students are confused! To make such a collaboration work, it is essential that group members trust each other enough to get past this initial sense of disbelief at each other's methods.

So what do we recommend to mathematics faculty teaching introductory courses? Here are some suggestions that we believe would help students apply calculus in their chosen discipline (including mathematics!). First of all, skip the fine print. In other words, emphasize examples, not counterexamples. Use numerical data in class, and discuss the implications. Ask students to determine derivatives experimentally. A good start is to have students actually *measure* rise over run from a graph. However, be sure to also include some examples that are not based on graphical information.

As indicated in [7], we encourage the use of infinitesimal reasoning, the art of working with quantities that are "small enough" for the purpose at hand. As we illustrated in a series of papers [6, 10, 15] and an online multivariable calculus text [14], differentials provide a robust, geometric, conceptual framework for working with such quantities; there are also others, such as power series. Differentials are often downplayed in single-variable calculus, despite their ubiquitous presence in "u-substitution." However, their importance in multivariable calculus (and in exact differential equations) justifies in our minds their inclusion right from the beginning. For instance, the standard symbolic differentiation and integration rules are all but identical when written in terms of differentials, and doing so leads to dramatic simplifications in the presentation of both chain rule and related rates [15].

All of these suggestions align well with the recommendations of the Curriculum Foundation Project of the MAA [17], which sought detailed input from partner disciplines: Emphasize conceptual understanding, problem solving skills, communication skills, and a balance between perspectives.

We have developed a variety of resources in order to implement these ideas in the classroom. First and foremost, the Portfolios Wiki [33] documents more than 300 small group activities for use across the physics major, indexed by topic, including both multivariable and vector calculus. We have written an accompanying online textbook [14], covering multivariable and vector calculus as well as applications to electromagnetism. All of these materials have been developed and used in the classroom over a period of 20 years. We have considerable qualitative data supporting the usefulness of this approach; see for example [21, 22]. Anecdotally, second-year calculus students and upper-division physics students and graduate TAs often ask "Why didn't we learn this before?" when introduced to differentials and infinitesimal reasoning, whereas first-year students, especially those who have had high-school calculus, often react with confusion when this language is used from the beginning.

Finally, we do not, of course, expect multivariable calculus courses to teach students how to solve problems in thermodynamics. Nonetheless, we do hope such courses prepare students to do so. Understanding derivatives, both ordinary and partial, as ratios of suitably small quantities, both in terms of infinitesimals and in terms of experimental data, provides a robust conceptual framework that we believe will allow students to apply calculus to science more easily.

ACKNOWLEDGEMENTS

Much of this work was done under the auspices of three overlapping projects. The Vector Calculus Bridge project [3, 9] seeks to bridge the gap between the way mathematicians teach vector calculus and the way physicists use it. The Paradigms in Physics project [26, 27, 33] has redesigned the entire upper-division physics curriculum at OSU, incorporating modern pedagogy and deep conceptual connections across traditional disciplinary boundaries; its website documents both the 18 new courses that resulted, and the more than 300 group activities that were developed. The Raising Calculus to the Surface [50] project uses plastic surfaces and accompanying contour maps, all dry erasable, to convey a geometric understanding of multivariable calculus. Figure 2 first appeared in [37]; Figure 1 is taken from the Surfaces project website [50] and is used with permission.

FUNDING

The Bridge and Paradigms projects have been supported by the NSF through grants DUE–9653250, 0088901, 0231032, 0231194, 0618877, 0837829, 1023120, and 1323800; the Surfaces project is supported by the NSF through grants DUE–1246094, 1611946, 1611970, and 1612480.

APPENDIX

Magnetic Susceptibility

We return to the example shown in Figure 5, and work out the details summarized in Section 5. We are given the magnetization M in terms of the magnetic field B and temperature T, and the temperature in terms of the magnetic field and the entropy S, so that

$$dM = \left(\frac{\partial M}{\partial B}\right)_T dB + \left(\frac{\partial M}{\partial T}\right)_B dT, \tag{10}$$

$$dT = \left(\frac{\partial T}{\partial B}\right)_S dB + \left(\frac{\partial T}{\partial S}\right)_B dS. \tag{11}$$

Substituting the second expression into the first yields

$$dM = \left(\frac{\partial M}{\partial B}\right)_T dB + \left(\frac{\partial M}{\partial T}\right)_B \left(\left(\frac{\partial T}{\partial B}\right)_S dB + \left(\frac{\partial T}{\partial S}\right)_B dS\right), \tag{12}$$

and comparison with

$$dM = \left(\frac{\partial M}{\partial B}\right)_S dB + \left(\frac{\partial M}{\partial S}\right)_B dS \tag{13}$$

(or simply setting $dS = 0$) leads immediately to equation (9). As noted in Section 5, this procedure is neatly summarized by either diagram in Figure 5, from which equation (9) can be read off without actually doing any computation.

In practice, these computations are often done with equations of state which give the relationships between the variables explicitly. An example used in the Paradigms program [33] has

$$M = N\mu \; \frac{e^{\frac{\mu B}{k_B T}} - e^{-\frac{\mu B}{k_B T}}}{e^{\frac{\mu B}{k_B T}} + e^{-\frac{\mu B}{k_B T}}}, \tag{14}$$

$$S = Nk_B \left[\ln 2 + \ln\left(e^{\frac{\mu B}{k_B T}} + e^{-\frac{\mu B}{k_B T}}\right) + \frac{\mu B}{k_B T} \frac{e^{\frac{\mu B}{k_B T}} - e^{-\frac{\mu B}{k_B T}}}{e^{\frac{\mu B}{k_B T}} + e^{-\frac{\mu B}{k_B T}}}\right]. \tag{15}$$

These expressions are not quite of the form shown in Figure 5, since S is given in terms of T (and B) rather than vice versa. However, since expressions involving differentials are linear, it is straightforward to perform the necessary rearrangements.

In this example, working directly with differentials reveals that both dM and dS are proportional to $(T\,dB - B\,dT)$, which should be obvious in retrospect, since both M and S are functions of the single variable B/T. Thus, setting $dS = 0$ results also in $dM = 0$, and the adiabatic magnetic susceptibility vanishes. (Not so the *isothermal* magnetic susceptibility, obtained by holding T constant rather than S.)

REFERENCES

1. Asiala, M., J. Cottrill, E. Dubinsky, and K. E. Schwingendorf. 1997. The development of students' graphical understanding of the derivative. *Journal of Mathematical Behavior.* 16: 399–431.
2. Berry, J. S. and M. A. Nyman. 2003. Promoting students' graphical understanding of the calculus. *Journal of Mathematical Behavior*. 22: 481–497.
3. The Vector Calculus Bridge Project. 2001-2019. http://math.oregonstate.edu/bridge. Accessed 11 May 2019.
4. Bucy, B. R., J. R. Thompson, and D. B. Mountcastle. 2007. Student (mis)application of partial differentiation to material properties. *AIP Conference Proceedings*. 883: 157–160.

5. Christensen, W. M. and J. R. Thompson. 2012. Investigating graphical representations of slope and derivative without a physics context. *Physics Review: Physics Education Research*. 8: 023021.
6. Dray, T. 2013. Using differentials to determine the derivatives of trigonometric and exponential functions. *College Mathematics Journal.* 44: 17–23.
7. Dray, T. May 2016. Thick derivatives. AMS Blog: On teaching and learning mathematics. http://blogs.ams.org/matheducation/2016/05/31/thick-derivatives. Accessed 11 May 2019.
8. Dray, T., B. Edwards, and C. A. Manogue. 2008. Bridging the gap between mathematics and physics. In M. Santos and Y. Shimizu (Eds), *Proceedings of the 11th International Congress on Mathematics Education*, ICME, Monterry, Mexico.
9. Dray, T. and C. A. Manogue. 1999. The vector calculus gap: Mathematics $\neq$ physics. *PRIMUS.* 9: 21–28.
10. Dray, T. and C. A. Manogue. 2003. Using differentials to bridge the vector calculus gap. *College Mathematics Journal.* 34: 283–290.
11. Dray, T. and C. A. Manogue. 2004. Bridging the gap between mathematics and physics. *APS Forum on Education*. Spring: 13–14.
12. Dray, T. and C. A. Manogue. 2005. Bridging the gap between mathematics and the physical sciences. In D. Smith and E. Swanson (Eds), *Preparing Future Science and Mathematics Teachers*, pp. 39–41. Bozeman, MT: Montana State University.
13. Dray, T. and C. A. Manogue. 2006. The geometry of the dot and cross products. *JOMA*. https://www.aps.org/units/fed/newsletters/spring2004/10oregonstate.html. Accessed 11 May 2019.
14. Dray, T. and C. A. Manogue. 2009–2016. The geometry of vector calculus. http://math.oregonstate.edu/BridgeBook. Accessed 11 May 2019.
15. Dray, T. and C. A. Manogue. 2010. Putting differentials back into calculus. *College Mathematics Journal.* 41: 90–100.
16. Emigh, P. J. and C. A. Manogue. 2017. Student interpretations of partial derivatives. In L. Ding, A. Traxler and Y. Cao (Eds), *Proceedings of the 2017 Physics Education Research Conference*. https://www.compadre.org/per/items/4761.pdf.
17. Ganter, S. and W. Barker. 2004. Curriculum foundations project: Voices of the partner disciplines. http://www.maa.org/the-curriculum-foundation-project-0. Accessed 11 May 2019.
18. Habre, S. and M. Abboud. 2006. Students' conceptual understanding of a function and its derivative in an experimental calculus course. *Journal of Mathematical Behavior.* 25: 57–72.
19. Hughes-Hallett, D., A. M. Gleason, W. G. McCallum, D. E. Flath, D. O. Lomen, D. Lovelock, J. Tecosky-Feldman, T. W. Tucker, J. Thrash, K. R. Rhea, A. Pasquale, S. P. Gordon, D. Quinney, and P. Frazer Lock. 1998. *Calculus: Single Variable, 2nd edition*. Hoboken, NJ: Wiley.
20. Jones, S .R. 2016. Ways of understanding and ways of thinking in using the derivative concept in applied (non-kinematic) contexts. In T. Fukawa-Connelly, N. E. Infante, M. Wawro, and S. Brown (Eds), *Proceedings of*

the 19th Annual Conference on Research in Undergraduate Mathematics Education, pp. 917–924. Washington, DC: MAA.

21. Kustusch, M. B., D. Roundy, T. Dray, and C. Manogue. 2012. An expert path through a thermo maze. *AIP Conference Proceedings*. 1513: 234–237.
22. Kustusch, M. B., D. Roundy, T. Dray, and C. A. Manogue. 2014. Partial derivative games in thermodynamics: A cognitive task analysis. *Physics Review: Physics Education Research.* 10: 010101. http://journals.aps.org/prstper/abstract/10.1103/PhysRevSTPER.10.010101. Accessed 11 May 2019.
23. Likwambe, B. and I. M. Christiansen. 2008. A case study of the development of in-service teachers' concept images of the derivative. *Pythagoras*. 68: 22–31.
24. Lithner, J. 2003. Students' mathematical reasoning in university textbook exercises. *Educational Studies in Mathematics.* 52(1): 29–55.
25. Manogue, C. and T. Dray. 2002. Functions. http://math.oregonstate.edu/bridge/ideas/functions. Accessed 11 May 2019.
26. Manogue, C. A. and K. S. Krane. 2003. The Oregon State University Paradigms Project: Re-envisioning the upper level. *Physics Today*. 56: 53–58.
27. Manogue, C. A., P. J. Siemens, J. Tate, K. Browne, M. L. Niess, and A. J. Wolfer. 2001. Paradigms in physics: A new upper-division curriculum. *American Journal of Physics.* 69: 978–990.
28. Marrongelle, K. A. 2004. How students use physics to reason about calculus tasks. *School Science and Mathematics*. 104: 258–272.
29. McCallum, W. G., D. Hughes-Hallett, D. E. Flath, B. G. Osgood, A. M. Gleason, D. Quinney, P. Frazer Lock, J. Tecosky-Feldman, D. Mumford, T. W. Tucker, S. P. Gordon, A. Iovita, W. Raskind, and and J. B. Thrash. 2005. *Calculus: Multivariable, 4th edition.*. Hoboken, NJ: Wiley.
30. Musgrave, S. and P. W. Thompson. 2014. Function notation as an idiom. In P. Liljedahl, S. Oesterle, C. Nicol, and D. Allan (Eds), *Proceedings of the 38th conference of the International Group for the Psychology of Mathematics Education*, volume 4, pp. 281–288. Vancouver, BC: PME.
31. Orton, A. 1983. Students' understanding of differentiation. *Educational Studies in Mathematics*, 14: 235–250.
32. Planinic, M., Z. Milin-Sipus, H. Katic, A. Susac, and L. Ivanjek. 2012. Comparison of student understanding of line graph slope in physics and mathematics. *International Journal of Science and Mathematics Education.* 10: 1393–1414.
33. Portfolios Wiki. 2007-2019. http://physics.oregonstate.edu/portfolioswiki.
34. Redish, E. and E. Kuo. 2015. Language of physics, language of math: Disciplinary culture and dynamic epistemology. *Science & Education.* 24: 561–590.
35. Roundy, D., T. Dray, C. A. Manogue, J. F. Wagner, and E. Weber. 2015. An extended theoretical framework for the concept of derivative. In T. Fukawa-Connelly, N. E. Infante, K. Keene, and M. Zandieh (Eds), *Proceedings of the 18th Annual Conference on Research in Undergraduate Mathematics Education*, 838–843. Washington, DC: MAA.

36. Roundy, D., M. B. Kustusch, and C. Manogue. 2014. Name the experiment! Interpreting thermodynamic derivatives as thought experiments. *American Journal of Physics*. 82(1): 39–46.
37. Roundy, D., E. Weber, T. Dray, R. R. Bajaracharya, A. Dorko, E. M. Smith, and C. A. Manogue. 2015. Experts' understanding of partial derivatives using the partial derivative machine. *Physics Review: Physics Education Research*. 11: 020126.
38. Sfard, A. 1991. On the dual nature of mathematical conceptions: Reflections on processes and objects as different sides of the same coin. *Educational Studies in Mathematics*. 22(1): 1–36.
39. Skemp, R. R. 1976. Relational understanding and instrumental understanding. *Mathematics Teaching*. 77: 20–26.
40. Styer, D. F. 1999. A thermodynamic derivative means an experiment. *American Journal of Physics*. 67(12): 1094–1095.
41. Thompson, J. R., B. R. Bucy, and D. B. Mountcastle. 2006. Assessing student understanding of partial derivatives in thermodynamics. *AIP Conference Proceedings*. 818: 77–80.
42. Thompson, P. W. 2013. In the absence of meaning. In K. Leatham (Ed.), *Vital Directions for Research in Mathematics Education*. New York, NY: Springer.
43. Thompson, P. W. and M. Ashbrook. 2016. Calculus: Newton meets technology. http://patthompson.net/ThompsonCalc. Accessed 11 May 2019.
44. Thompson, P. W., C. Byerley, and N. Hatfield. 2013. A conceptual approach to calculus made possible by technology. *Computers in the Schools*. 30: 124–147.
45. Thompson, P. W. and T. Dreyfus. 2016. A coherent approach to the Fundamental Theorem of Calculus using differentials. In R. Göller, R. Biehler, and R. Hochsmith (Eds), *Proceedings of the Conference on Didactics of Mathematics in Higher Education in Scientific Disciplines*, pp. 355–359. Hannover, Germany: KHDM.
46. Tucker, T. W. (Ed.). 1990 *Priming the Calculus Pump: Innovations and Resources*, Washington, DC: MAA.
47. Van Zee, E, and C. Manogue. 2018. A study of the development of the Paradigms in Physics Program. http://physics.oregonstate.edu/mathbook/P20/. Accessed 11 May 2019.
48. Vinner, S. 1983. Concept definition, concept image and the notion of function. *International Journal of Mathematical Education in Science and Technology.* 14: 293–305.
49. Wangberg, A. 2012. Raising students' calculus understandings to the surface in multivariable calculus. In S. Brown, S. Larsen, K. Marrongelle, and M. Oehrtman (Eds), *Proceedings of the 15th Annual Conference on Research in Undergraduate Mathematics Education*, volume 2, pp. 590–594. Washington, DC: MAA.
50. Wangberg, A. 2013–2019. Raising Calculus to the Surface. http://raisingcalculus.winona.edu. Accessed 11 May 2019.
51. Wangberg, A. and B. Johnson. 2013. Discovering calculus on the surface. *PRIMUS.* 23: 627–639.

52. Zandieh, M. 2000. A theoretical framework for analyzing student understanding of the concept of derivative. *CBMS Issues in Mathematics Education*, 8: 103–122.

BIOGRAPHICAL SKETCHES

Tevian Dray, professor of mathematics at Oregon State University, is a geometer with a longstanding interest in the interface between mathematics and physics, as well as in mathematics education. He has been the director of the Vector Calculus Bridge project since 2001, a co-PI on the Paradigms project since its inception in 1996, and has also participated in several projects aimed at improving the content knowledge of mathematics teachers. The underlying theme in both his traditional research in mathematical physics and his work in education has been the importance of geometric reasoning.

Elizabeth Gire does research in physics education, with emphasis on problem-solving and the development of epistemological beliefs and metacognitive skills of undergraduate physics majors. She worked on the Paradigms project as a postdoctoral Research Associate from 2007–2009, where she taught and conducted research in the context of the Paradigms courses. She has recently returned to Oregon State University as an assistant professor of physics at Oregon State University, and is a co-PI on the Paradigms project.

Mary Bridget Kustusch, assistant professor of physics at DePaul University, specializes in physics education research. As a postdoctoral scholar with the Paradigms project from 2011–2013, she conducted some of the initial research on expert use of partial derivatives in thermodynamics. Her current research continues to explore the development of scientific expertise and identity, although now with more of a focus on the impact of group interactions in the classroom. In addition to implementing many of the pedagogical approaches from the Bridge and Paradigms projects in her own classes, she has conducted local workshops on embodied learning activities and the use of whiteboards for promoting student discourse.

Corinne A. Manogue, professor of physics at Oregon State University, has directed the Paradigms project from the beginning. She has 20 years experience not only developing and teaching multiple courses in the Paradigms program, but also working with the entire Paradigms team to create a coherent curriculum characterized by active learning. Her special interest, investigating the role that active-engagement experiences play in helping students transition from lower-division mathematics to upper-division physics, led to her involvement as co-PI in the

Vector Calculus Bridge project. Her traditional research in theoretical quantum gravity uses the octonions to describe the symmetries of high energy particle physics.

David Roundy, associate professor of physics at Oregon State University, has been co-PI on the Paradigms project since 2010. He does research in computational condensed matter physics, and his curriculum development has focused on improving the teaching of computational and thermal physics.