Using Differentials in Thermodynamics

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Thermodynamics provides a powerful context in which to explore expert and student understanding of partial derivatives, differentials, and chain rules. We summarize here our efforts to identify expert reasoning in this arena and convey it to students as part of the Paradigms in Physics project at Oregon State University, now in its third decade. In particular, we describe expert use of differentials to manipulate partial derivatives and analyze student difficulties in moving between different representations of partial derivatives.

Keywords: Teaching and learning of specific topics in calculus, teachers' and students' practices related to calculus across disciplines, differentials, partial derivatives, thermodynamics.

INTRODUCTION

A typical question in thermodynamics is to determine the *adiabatic bulk modulus*, defined by $\beta_S = -V\left(\frac{\partial p}{\partial V}\right)_S$, which measures resistance to compression [1]. For example, the entropy S might be given by

$$S = Nk\left(\ln\left[\frac{V}{N}\left(\frac{mkT}{2\pi\hbar^2}\right)^{3/2}\right] + \frac{5}{2}\right)$$

in terms of the volume V and the temperature T, and for an ideal gas we would have the *equation of state* $pV = NkT$, relating the pressure p, volume V, and temperature T; where $\hbar$, N, m, and k are constants. It is not immediately obvious how to determine the desired partial derivative from the given information, nor which variables are independent—or even how many there are.

We summarize here some of our efforts to investigate understanding of partial derivatives, including a cognitive task analysis of expert approaches to problems such as the one above and thematic analyses of student difficulties in applying their mathematical knowledge to such problems. This work is part of the Paradigms in Physics project at Oregon State University, which for nearly 30 years has reimagined the undergraduate physics major, not only incorporating and adapting modern pedagogical strategies, but also significantly rearranging the content, based on the education research of ourselves and others.

SOLVING THE PROBLEM WITH DIFFERENTIALS

We begin with a task analysis of the underlying mathematics. The geometric approach to calculus used in the Paradigms project emphasizes infinitesimal reasoning using *differentials* to represent quantities that are "small enough" to model linear differential relationships to the desired accuracy (Dray & Manogue, 2003, 2010; Dray, 2016, Dray et al., 2019). We teach our students to "zap with *d*", converting an equation such as $xy = 1$ to $x\,dy + y\,dx = 0$. This process automatically keeps track of which derivatives have been taken; it is not necessary to decide beforehand which variables are independent.

In the example above, zapping the equation for S with d and a little algebra yields

$$dS = \frac{3Nk}{2T}dT + \frac{Nk}{V}dV$$

and setting this expression equal to zero (since S is constant) yields a *linear* relation between dT and dV, which can of course be solved for either differential. Zapping the ideal gas law with d and eliminating dT leads directly to $\beta_S = \frac{5NkT}{3V}$.

USING CHAIN RULE DIAGRAMS TO KEEP TRACK

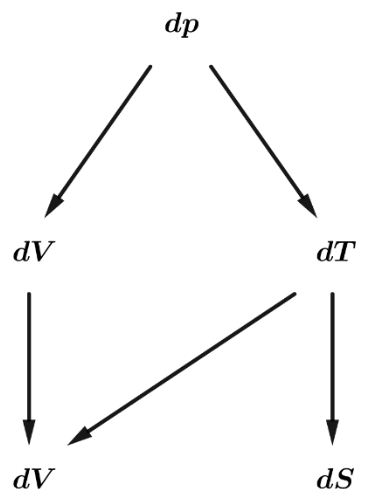

Figure 1: A chain rule diagram for the calculation of adiabatic bulk modulus. Arrows indicate that the upper differential depends (linearly!) on the lower differential.

One great advantage of the differentials approach is that it is always possible to solve linear equations involving differentials, unlike the equations for the original variables. One disadvantage of this strategy is that it is easy to lose track of where in the calculation one is. We encourage students to use *chain rule diagrams* such as Figure 1 as a reminder of which partial derivatives are needed. Similar diagrams are often used in mathematics textbooks to represent the multivariable chain rule, although we prefer to write such diagrams directly in terms of differentials. The diagram in Figure 1 is equivalent to the chain rule expression

$$\left(\frac{\partial p}{\partial V}\right)_S = \left(\frac{\partial p}{\partial V}\right)_T + \left(\frac{\partial p}{\partial T}\right)_V \left(\frac{\partial T}{\partial V}\right)_S .$$

WHAT OUR RESEARCH SHOWED

Expert Reasoning

Kustusch et al. (2012, 2014) interviewed ten experts in several disciplines while giving them a problem much like the example above. No two experts approached the problem the same way. Three basic strategies, or "epistemic games", were identified using cognitive task analysis: using *substitution* to isolate the independent variable prior to differentiation, using various forms of the multivariable chain rule to relate *partial derivatives,* and using *differentials* to reduce the problem to linear algebra. Substitution was seen to be problematic if the given equations are difficult to solve, and partial derivatives and differentials were described by one expert as encoding the same information quite differently. Interestingly, some of the experts mentioned their concern that some of the (correct) moves that they made would not be considered "legal" by mathematicians, especially in the context of working with differentials. This research prompted the development of the "Partial Derivatives Machine" (PDM), a simple mechanical device with springs and pulleys that provides an exact mathematical analogue to classical thermodynamic systems such as gas in pistons. We have developed curriculum around the PDM and done some initial studies on its effectiveness in the classroom (Roundy et al., 2015; Paradigms Team, 2015–2024).

Student Reasoning

Founds et al. (2017) used an emergent coding scheme to identify and categorize the solution methods of physics students in the Paradigms program on analogous chain rule problems from both a pure algebra question on a quiz (N=29) and then a thermodynamics inspired question that was part of the final exam (N=27). The study examined both what solution strategies students chose to employ and what types of conceptual errors they made. This analysis used a more finely grained classification of the solution strategies, namely *variable substitution, differential substitution, implicit differentiation, differential division, chain rule diagrams,* as well as several strategies that did not lead to an answer. Each of these strategies except the first had been explicitly discussed in class, with emphasis on differential substitution and chain rule diagrams and the quiz was reviewed in class before the final exam.

Many students (31%) attempted to use the familiar technique of variable substitution on the (much easier) quiz problem, where it was indeed a viable solution strategy, but fewer (only 11%) on the exam. It is not clear whether students recognized that the exam computation would be extremely lengthy, or whether they had mastered another technique. No students made conceptual errors with this technique. Differential substitution was used by more students on the exam (44%) than on the quiz (21%) and by the time of the exam no students made conceptual errors with this method. Chain rule diagrams were also relatively common, (21%) on the quiz and (22%) on the final. While several students made conceptual errors in their chain rule diagrams on the quiz (building incorrect diagrams and/or misreading them), none made such errors on the exam. Other methods were less common.

In a follow-up study of 12 students, Founds & Manogue (2022) found, using thematic analysis, that the difficulties many junior-level physics students experience may be related to their unfamiliarity with Leibniz notation. In addition, this study showed that these students do not know to eliminate extra dependent variables in systems of equations.

Multiple Representations

In a separate study, Bajracharya et al. (2019) asked eight student interviewees a more difficult prompt: to determine a particular partial derivative from data with some presented graphically and other data presented numerically in a table. To solve this problem successfully, interviewees not only needed to derive a chain rule analogous to the one above, but also to identify which partial derivative could be found from the data as presented. This study introduced *representational transformation diagrams* as a method to describe student problem-solving strategies. These strategies included both graphical analysis and analytic derivations using tools such as differentials and tree diagrams. The analysis focused on students' ability to transform one representation into another, identifying several classes of transformations such as *translation*, *consolidation*, and *dissociation*. Consolidation, a process in which a student transforms two or more representations into a single representation, and dissociation, a process in which one representation is expanded into two or more representations, were found to be the most common places for interviewees to encounter difficulties.

SUMMARY

These results document the difficulties some students have based on their mathematical training when trying to master the expert reasoning around (partial) derivatives used in physics. They also document the existence of several expert approaches to the same task, both across disciplines and within a single discipline. Helping students become experts will require interdisciplinary coordination.

NOTES

1. This common generalization of Leibniz notation for partial derivatives in thermodynamics will be unfamiliar to many mathematicians and most students, with the subscript indicating the derivative *with the entropy S held fixed*.

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