Letter to the Editor:

Finding Geodesics

We have several comments on the article "A Difficult Climb" in the February issue (*Math Horizons*, Feb. 2009, pp. 12-15, 29).

Units

The article gives the height of the hill in feet as $h(x, y) = 5000 - 30x^2 - 10y^2$, with x and y in miles, but then computes the arc lengths as if the vertical units were miles. If h really were in feet, then the (initial) slope of the hill from (1, 4) to the top would be barely 1°, while if all units are taken to be miles, then the slope becomes 89°, a possibility for a skilled rock climber. Adopting the conventions in the article, the direct path would in fact be 22,231 feet long, whereas the gradient path would be 21,771 feet long. Here, we will take the vertical distance to be in miles, as otherwise the hill is almost flat and the geodesic (shortest) path turns out to be very close to the direct path.

The Geodesic

The shortest path can be determined by solving the geodesic equation

$$\mathbf{X}'' = -\nabla h \frac{(\nabla h)' \cdot \mathbf{X}'}{1 + \|\nabla h\|^2},$$

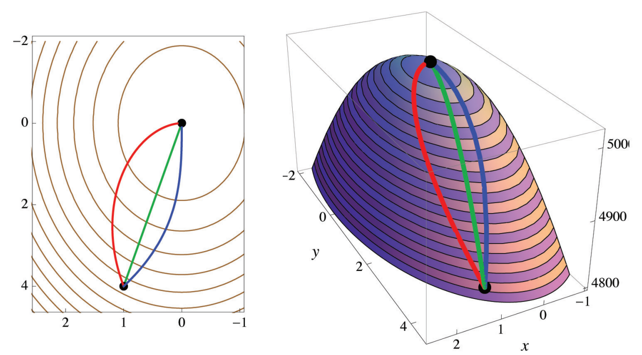
with given endpoints where $\mathbf{X} = x\mathbf{i} + y\mathbf{j}$ and prime denotes differentiation with respect to the arc length parameter s. This equation can be obtained either using the techniques of differential geometry (a "straight" line is one whose "direction" doesn't change, as explained at `http://en.wikipedia.org/wiki/Geodesic`) or using a variational principle (a "straight" line minimizes the distance between nearby points, as explained at `http://mathworld.wolfram.com/Geodesic.html`). For the hill of the article, these equations become

$$\mathbf{X}'' = -4 \frac{\left(a(x')^2 + b(y')^2\right)}{\left(1 + 4a^2x^2 + 4b^2y^2\right)} (ax\mathbf{i} + by\mathbf{j})$$

where $a = 30$ and $b = 10$. As in the article, we take the initial point to be $\mathbf{X}(0) = \mathbf{i} + 4\mathbf{j}$ and the final point to be the top of the hill, scaling s so that $\mathbf{X}(1) = 0\mathbf{i} + 0\mathbf{j}$.

The `NDSolve` command in *Mathematica* can in principle solve this boundary-value problem numerically, but there are complications because such problems are not as simple as a standard initial-value problem. However, one can assist the algorithm by guessing the starting direction of the geodesic, and some experimentation with that choice leads to the true geodesic. One can then confirm the result by using higher precision and observing that the error decreases.

Below is a contour plot showing the geodesic obtained by `NDSolve`. The geodesic length is 190.086 miles (compared to 190.139 miles for the gradient path and 190.128 miles for the direct path, as given in the article). Showing the surface in true scale is not realistic because of the huge vertical spread, so we compress the vertical scale in the accompanying image to show the geodesic (red), the gradient path (blue), and the direct path (green).



The Conjecture

In the article, Sal conjectured that the shortest path must somewhere follow the gradient direction. It is clear from the figures above that this conjecture is false.

Stuart Boersma, Central Washington University, Ellensburg, WA
Tevian Dray, Oregon State University, Corvallis, OR
Stan Wagon, Macalester College, St. Paul, MN

The Mathematica notebook used to generate the figures above, which includes a discussion of the underlying code, is available on the Math Horizons website `www.maa.org/mathhorizons/supplemental.htm`. ■

10.4169/194762109X472053