> **“What are we trying to do again? Find the shortest path, the quickest path, or the steepest path to the top?”**

A Difficult Climb

or Dialogue Concerning the Two Chief Paths

Stuart Boersma
Central Washington University

[*Scene: Two students, Simplicio and Sal, meet on a college campus*]

Simplicio: Greetings Sal! I wanted to tell you about a problem from my calculus class. Maybe you worked on the same one in your class?

Sal: Did it involve climbing a hill?

Simplicio: Yes, that's the one! Didn't you find it raised many interesting geometry questions?

Sal: No, not really. It seemed pretty cut and dried to me. In fact, our group finished it well before class was over.

Simplicio: Oh. Did you find the steepest path to the top of the hill?

Sal: Yes.

Simplicio: Did you find the shortest path to the top of the hill?

Sal: Of course!

Simplicio: Didn't these answers conflict with each other and your intuition?

Sal: Not at all!

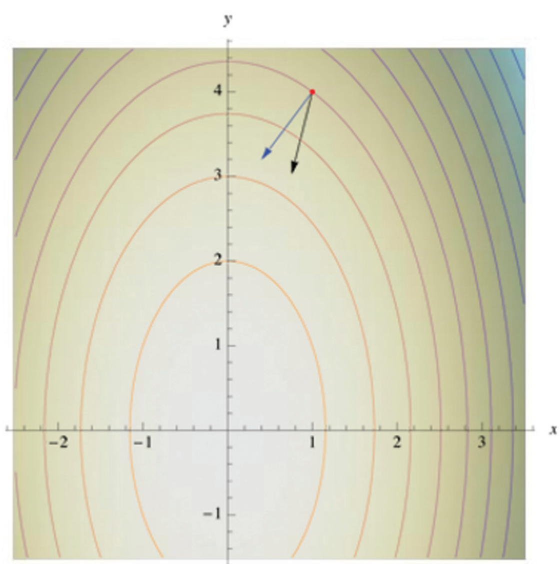

Figure 1. Two unit vectors drawn at (1,4). The black vector, $\vec{v}_1$, points directly towards the origin; the blue vector, $\vec{v}_2$, is perpendicular to the level curve passing though (1,4).

Simplicio: Okay, I guess you'll need to explain it all to me. In our problem we were supposed to assume we were out hiking on a hill whose height in feet above sea level was given by the function $h(x,y) = 5000 - 30x^2 - 10y^2$. The constants 30 and 10 had units so that x and y carried the units of miles.

Sal: That sounds similar to the problem we worked on. I think our instructor changed some of the numbers.

Simplicio: Okay, then the first item of business was to construct a topographic map of the hill. This is just a contour diagram, so it wasn't too difficult. Just a bunch of concentric ellipses surrounding the top of the hill (the origin).

Sal: Our contour diagram was composed of concentric circles. But otherwise, yes we got the same sort of picture.

Simplicio: First our instructor told us we were to assume we were located at position (1,4) on the map. After that we were asked to identify a direction in the topo map which would indicate which direction we should travel if we wanted to climb the hill as steeply as possible.

Sal: Yes, yes, I think you had the same assignment we did.

Simplicio: Our group decided that this was quite straight-forward. We then proceeded to quickly sketch in **two** different unit vectors! One pointed directly to the top of the hill,

$$\vec{v}_1 = \frac{1}{\sqrt{17}}\left(-\hat{i} - 4\hat{j}\right),$$

and the other was drawn so that it would be perpendicular to the level curve which went through the point (1,4). Since the gradient vector points in such a direction,

$$\vec{\nabla} h = -60\hat{i} - 80\hat{j},$$

we simply defined $\vec{v}_2$ to be a unit vector pointing in the same direction as $\vec{\nabla} h$ and sketched it along with $\vec{v}_1$. Just from the contour diagram you can tell that this vector doesn't point to the top of the hill. It misses to the west!

Sal: Aaah! I'm beginning to see why you ran into questions! If your level curves were circles as in our assignment, these two vectors would point in the same direction and no disagreements would erupt. So, what did your group decide? Did you simply use your common sense, as it is clear that if you wanted to get to the top of the hill the quickest you should start waking straight towards the top. Or did you decide that the rigor of calculus (or at least the intimidation of your professor!) should override common sense and you should start off in the direction of the gradient vector?

Simplicio: Well, when we were sure that the professor was out of earshot, we decided that the best course of action would be to construct the solution we believed our professor wanted. After all, this assignment was going to be graded!!! Since our instructor had, just the previous day, made a big deal about gradient vectors pointing in directions of greatest increase, we thought we had better follow this approach as we wrote up our solution.

Sal: Good thinking Simplicio!! However, now that class is over and your assignment has been turned in, we can focus our attention on which solution would, in fact, be the correct one.

Simplicio: Obviously we just ignore this crazy subject of calculus and just follow the vector $\vec{v}_1$ directly to the top of the hill.

Sal: Why obviously?

Simplicio: If we follow the gradient vector we don't even get to the top!

Sal: Hmmmm... Something isn't right here. According to your diagram, if I follow your gradient vector I will climb the hill for a while and then begin to descend without ever crossing the top. Clearly, that can't be right. Gradient vectors should at least always point up!

Simplicio: ....

Sal: Look here! Even though the gradient vector is perpendicular to the level curve at its base, out at the tip it is no longer perpendicular! Yes, that's it! You graphed the gradient vector at the point (1,4). Once you leave that point, the gradient vector will change! You need to continually re-plot your gradient vector so that you will always be traveling perpendicular to the level curves. We can easily sketch such a curve.... So you end up approaching the top of the hill along the positive y-axis.

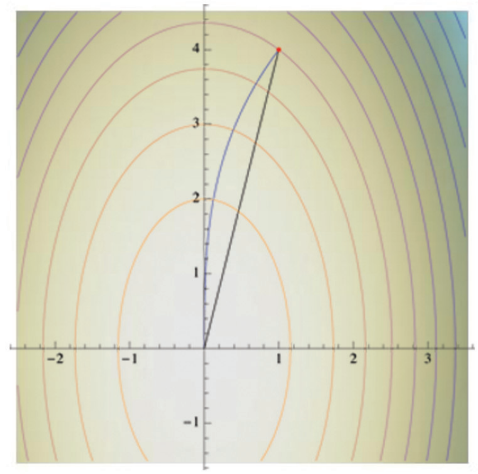

Figure 2. Two approaches to the summit. The blue curve is the gradient path. It proceeds in the direction indicated by the gradient vector, and is hence perpendicular to every level curve. The black curve follows a straight path.

Following the gradient vector certainly gets you to the top of the hill! (See Figure 2.)

Simplicio: But the path you drew, let's call it the *gradient path*, is apparently longer than my path. I'll grant you that the gradient path makes it to the top, but how can you argue that it is the steepest path if it doesn't get you to the top the quickest?

Sal: Now I'm confused. What are we trying to do again? Find the shortest path to the top? Find the quickest path to the top? Or, find the steepest path to the top?

Simplicio: Well, it seems to me that if you are climbing the hill as steeply as possible at every possible moment, then you must be gaining elevation at the quickest possible rate which in turn should guarantee reaching the summit in the shortest amount of time. If we further assume that as hikers we are very fit and are not winded by steep trails, the quickest route to the top should then be equivalent to the shortest path. So all three problems: shortest, quickest, and steepest are one and the same.

Sal: Hmmmm...It should be simple to check. We learned last semester how to compute lengths of curves, so all we have to do is compute the length of these curves to see which is shortest!

Simplicio: But, isn't it already clear that the straight-line is shorter than the curved path (the gradient path)?

Sal: On the topo map it's clear. But let's not forget that we are actually hiking along a surface! If you draw straight-lines on two-dimensional maps, they do not always translate into the shortest path in real-life (i.e. on a globe). Just think about flying from New York to Madrid. Even though they lie on (about) the same line of latitude, a flight is sure to deviate towards the North Pole.

Simplicio: Okay, let's do the computation. If I start at the point $(1,4)$ and travel directly to the origin, I can parameterize the path by:

$$\begin{aligned}\vec{r}(t) &= 1(1-t)\hat{i} + 4(1-t)\hat{j} + h(x(t),y(t))\hat{k}\\ &= 1(1-t)\hat{i} + 4(1-t)\hat{j} + (5000 - 30(1-t)^2 - 10(4(1-t))^2)\hat{k}\end{aligned}$$

where $0 \le t \le 1$. Now, it's length is given by

$$\begin{aligned}s &= \int_0^1 \sqrt{\frac{dx}{dt}^2 + \frac{dy}{dt}^2 + \frac{dz}{dt}^2}\, dt\\ &= \int_0^1 \sqrt{1 + 16 + (-380t + 380)^2}\, dt\\ &\approx 190.128\end{aligned}$$

Sal: Okay, now how about the gradient path?

Simplicio: Well, we need a curve $\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j}$ so that $\frac{d\vec{r}}{dt}$ is always parallel with $\vec{\nabla} h(x(t),y(t))$.

Sal: So we need:

$$\begin{aligned}x'(t) &= -60x(t)\\ y'(t) &= -20y(t).\end{aligned}$$

This just means $x(t)$ and $y(t)$ are exponential functions! If we want to start at the point (1,4) we can let $0 \le t < \infty$ and

$$\begin{aligned}x(t) &= e^{-60t}\\ y(t) &= 4e^{-20t}.\end{aligned}$$

Along the hill we have our gradient path:

$$\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j} + h(x(t),y(t))\hat{k}$$

Using the same formula for arc-length yields a length of 190.139. A little longer! Although it is so close to the first length we calculated that we may start to worry about the accuracy of our numerical approximations.

Simplicio: I'm willing to believe *Maple* for now, as the answer does mesh with my intuition that the gradient path is the longer path. Thus, the straight line path is the shortest.

Sal: Of the two.

Simplicio: ???

Sal: Maybe there is a third path that is shorter yet. Putting that aside for the moment, I still have a question. How do we reconcile the fact that the gradient vector is supposed to show us the steepest way up the hill, but it doesn't get us there in the shortest distance?

Simplicio: I think we may need some outside help here. Oh, we're in luck, here comes Professor Remawn. Professor Remawn, do you have a moment to spare to help us out of a mathematical dilemma?

Remawn: Yes, just don't let me forget which way I'm heading. I'd hate to miss lunch again.
[*Simplicio and Sal give Remawn an explanation of what they have so far*]

Simplicio: So, how is it that the gradient vector points in the steepest direction, but by continually following the gradient vector we end up **not** climbing the hill in the steepest (i.e. quickest) possible manner?

Remawn: Well, let's see. Initially, as you are standing at the point $(1,4)$ eagerly anticipating that first step towards the top of the hill do you agree that $\vec{\nabla} h$ points in the steepest direction?

Simplicio: Everything we've done in calculus would seem to indicate this. Also, I'm pretty sure that if I computed the appropriate directional derivatives of h we could verify that the greatest increase would be in the direction of $\vec{\nabla} h$.

Remawn: Fine. Now just keep in mind that the properties about gradients that you've been studying are *local* properties. They really don't say anything about the global nature of the problem. I hope this helps. Now which way was I going?

Sal: [*shooing away a fly*] Aaahh...

Remawn: Great. That means I've already had lunch. Carry on! [*begins to walk away, saying over his shoulder*] And remember, being greedy early on may have drastic consequences later. [*walks away*]

Simplicio: Professor Remawn! Wait! Oh well, I suppose he could stand to lose a few pounds anyway. Did you understand what he said?

Sal: I think so. Just because that first step is the steepest doesn't necessarily guarantee that the second and third and fourth step will be steeper compared to the hiker walking directly towards the top. In fact, from our contour diagram, it is clear that the hiker following the gradient path will eventually end up approaching the top of the hill from the *least* steep direction (the north)!

Simplicio: I do much better graphically. Let's get *Maple* to produce "steepness" plots for each path. For each path, we'll plot the directional derivative of h in the direction of travel. Look! (See Figure 3.)

Sal: That pretty much takes care of things. The gradient path starts off much steeper, but as it takes you to the north side of the hill, it loses steepness and so, in the long run, really isn't that steep of a path.

Simplicio: What about (4,1)?

Sal: Huh?

Simplicio: What if we started instead at the

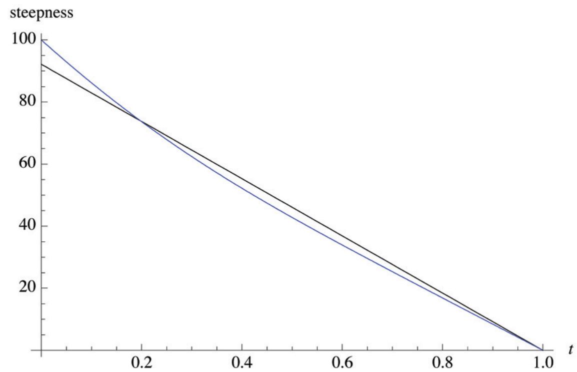

Figure 3. Plot of the "steepness" of the two approaches, parameterized by a variable *t* which is the proportion of the distance traveled along the path from the point (1, 4) to the origin. The gray curve represents values of the directional derivative of $h(x,y)$ as one travels directly towards the origin while the blue curve represents values of the directional derivative of $h(x,y)$ as one travels along the gradient path. The blue curve starts higher but gradually becomes less "steep."

point (4,1)? In the topo map it is the same distance from the origin as (1,4), but if you start there and follow the gradient path, it appears (from the contour diagram) that you end up approaching the top of the hill from the *steepest* part of the hill!

Sal: No, I don't think so. In fact, if you start anywhere in the first quadrant, say (a,b) then the gradient path can be parameterized by $\vec{r}(t) = ae^{-60t}\hat{i} + be^{-20t}\hat{j}$ which means that the tangent line to this curve has slope

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy/dt}{dx/dt}\\ &= \frac{-20b\,e^{-20t}}{-60a\,e^{-60t}}\\ &= \frac{b}{3a}e^{40t} \to \infty \quad \text{as} \quad t \to \infty.\end{aligned}$$

So, every gradient path (in the first quadrant) approaches the origin in such a way that its slope is infinite. That is, it approaches the origin tangent to the positive y-axis!

Simplicio: I guess Professor Remawn was right. Following the gradient path (the greedy path) means that as you arrive at the top, you are arriving in the least-steep manner possible! (Since the level curves are furthest apart in the y-direction!) So, even though following the gradient path assures you that each step is the steepest possible step at that time, in the long run it doesn't do a very good job of getting to the top in the steepest possible way!

Sal: Another interesting paradox is to notice that in following the straight line path, which gets one to the top quicker than the gradient path, one is **never** traveling in a direction parallel to the gradient vector! So, by never taking the steepest possible step, one achieves an overall steeper path!

Simplicio: I feel much better now that we've answered all of the pertinent questions.

Sal: What! We have a long way to go! If the gradient path doesn't get you to the top the quickest, then what is it good for? Is there a path which gets to the top quicker than even the straight-line path? In fact, maybe I'll go out on limb here and make a conjecture.

Simplicio: Now, you're scaring me. You sound too much like a mathematician. What is your conjecture?

Sal: Well, I mentioned just a minute ago that the straight-line path never moves in a direction parallel to the gradient vector anywhere along the path. It seems to me that if one did find a path that reached the top of the hill in the quickest possible manner, then at least *at some point* one should be traveling in the direction of the gradient vector at that point. That is, one should take the optimal step in terms of gaining elevation at least once!

Simplicio: I don't know. I'm beginning to believe that one's intuition isn't good for much as far as this problem is concerned. In any case, I have no idea how one would answer, or even test, such a conjecture. How do you find the overall quickest route up the hill? Maybe Professor Monge can help. I see him falling out of his classroom window now. Let's ask.
[*both run over to see that Professor Monge is unhurt*]

Continued on page 29

Continued from page 15

Sal: Professor Monge, is it possible to find the shortest path up a surface?

Monge: Hmm... What...? Oh. Shortest path? Yes, yes. Done all the time. Airplanes, geodesics, and all that.

Sal: Great! How do you do it?

Monge: Do what?

Sal: Calculate geodesics.

Monge: Oh... Well... Let's see... You could take a *Calculus of Variations* approach and minimize the integral that represents the arc-length between two given points. You'll have to be sure to minimize this integral over all possible smooth paths that begin and end at the given points. Or, you could use your surface to construct coefficients for the metric tensor and the Christoffel symbols and set up the geodesic equation. Of course, the result can be a fairly complicated system of differential equations. But given some decent assumptions you might have a crack at it. All good fun, geodesics are. Now, I had better get back to class. [*begins climbing back through the open window*]

Sal: Hmmm... Maybe we better keep taking math classes to find out what this is all about.

Simplicio: Or, we can just plod along and trust our common sense.

TO BE CONTINUED… at your campus! Ask your professors about differential geometry, geodesics, and curvature! ■

Acknowledgement

The original student activity which generated this discussion between Simplicio and Sal is due to Professors Tevian Dray and Corinne Manogue from Oregon State University.