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# Using Embedding Diagrams to Visualize Curvature

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**Abstract.** We give an elementary treatment of the curvature of surfaces of revolution in the language of vector calculus, using differentials rather than an explicit parameterization. We illustrate some basic features of curvature using embedding diagrams, and then use such a diagram to analyze the geometry of the Schwarzschild black hole.

**1. INTRODUCTION.** Visualizing surfaces in three dimensions is, for most of us, an acquired skill. One of the steps along the way is to develop an understanding of a surface of revolution, which can be represented in terms of its *generating curve*, showing a cross-section of the surface, with the angular symmetry suppressed. Most such surfaces have nonzero (Gaussian) curvature, but what does this mean, and what is the relationship between curvature and generating curves?

General relativity is the study of curved, four-dimensional spacetimes (Lorentzian manifolds), and especially their (Riemann) curvature. Again, most physical spacetimes have nonzero curvature, yet even in the presence of symmetry it is not apparent what this means. One technique for visualizing the curvature of spacetime is to study the curvature of three-dimensional spacelike cross-sections, that is, surfaces of constant “time.” In such cases, the *intrinsic* geometry of these surfaces represents physical space, whose curvature can be visualized by an embedding into four-dimensional Euclidean space. The resulting *embedding diagrams* are higher-dimensional analogs of generating curves, normally drawn with at least one dimension suppressed.

It is straightforward to investigate the curvature of a given surface of revolution, since the induced line element (metric tensor) can easily be computed. In relativity, one typically works in the other direction; only the line element is known, but not the corresponding surface of revolution (if any). Although the curvature can easily be computed, the question is how to interpret the result.

We present here an analysis of (some) surfaces of revolution with these goals in mind, both finding the curvature and determining the shape to which it corresponds.

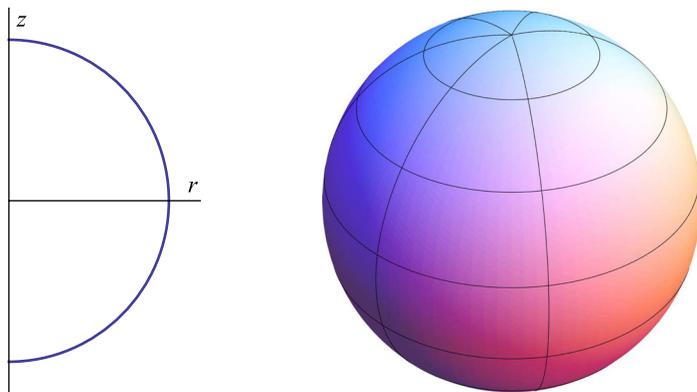
Although the curvature of curves is usually covered in a multivariable calculus class [2, 20], the generalization to surfaces appears in the undergraduate curriculum, if at all, in a course on differential geometry focusing on curves and surfaces in Euclidean space. Many such treatments use parametric curves and surfaces [1, 5]. However, most treatments of general relativity, even those aimed at undergraduates, use tensor analysis, although some more recent texts, such as [14], delay the introduction of this machinery as long as possible.

We describe here a slightly different approach, emphasizing the use of differentials [8] and a “use what you know” philosophy [7]. We believe this approach strikes a useful balance between the skills learned in calculus and the use of tensor analysis, allowing easy access to sophisticated reasoning as well as a path to the mastery of differential forms, yet without emphasizing parameterization. The examples here are in fact adapted from just such a course, a two-term (20-week) introduction to differential forms and general relativity that has been running successfully for 20 years [6].

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**Figure 1.** A vertical cross-section (left) of a sphere (right), in cylindrical coordinates.

We begin in [Section 2](#) with a motivating example, the sphere, followed by a discussion of the plane, cylinder, and cone in [Section 3](#). To complete our tour of surfaces of constant curvature, we then analyze the pseudosphere in [Section 4](#), before giving a general treatment of surfaces of revolution in [Section 5](#). Finally, in [Section 6](#), we analyze  $t = \text{constant}$  slices of the Schwarzschild geometry, the unique, spherically-symmetric, vacuum solution of Einstein’s field equation, now known to describe a black hole. In this case, our first task is to determine the shape of the generating curve from the line element, after which we can compute the (Gaussian) curvature of the slice. The behavior of this curvature along the slice turns out to provide several key hints as to the unexpected features of this geometry.

We emphasize that none of our results are new, although our treatment is somewhat nonstandard, and our choice of examples somewhat eclectic.

**2. MOTIVATING EXAMPLE: THE SPHERE.** Surfaces of revolution are conveniently described in cylindrical coordinates  $(r, \phi, z)$ ; their generating curves can be regarded as curves in the two-dimensional half-plane with  $\phi = 0$ , henceforth referred to as the  $rz$ -plane. Consider the semicircle in the  $rz$ -plane with equation  $r^2 + z^2 = a^2$  (and  $\phi = 0$ ), as shown in [Figure 1](#). A point on this curve can be described using the position vector  $\vec{r} = r \hat{r} + z \hat{z}$ , where hats denote unit vectors in the direction in which the corresponding coordinate increases. Treating  $z$  as a function of  $r$ , the unit tangent vector to this circle is implicitly defined by  $\hat{T} ds = d\vec{r}$ , where

$$ds^2 = d\vec{r} \cdot d\vec{r} = (dr \hat{r} + dz \hat{z}) \cdot (dr \hat{r} + dz \hat{z}) = dr^2 + dz^2, \quad (1)$$

where we have used the fact that  $\hat{r}$  and  $\hat{z}$  are constant in the  $rz$ -plane. The differential version of  $r^2 + z^2 = a^2$  can be computed using implicit differentiation, resulting in  $r dr + z dz = 0$ . Inserting this relation into (1), we have

$$ds^2 = \left(1 + \frac{r^2}{z^2}\right) dr^2 = \frac{a^2}{z^2} dr^2 = \frac{a^2 dr^2}{a^2 - r^2}, \quad (2)$$

so that  $ds = (a/z) dr$ , which implies that

$$\hat{T} = \frac{\hat{r} - r/z \hat{z}}{a/z} = \frac{z \hat{r} - r \hat{z}}{a}. \quad (3)$$

The *curvature*  $\kappa$  and *principal unit normal vector*  $\hat{N}$  of a curve, as introduced in multivariable calculus [2, 20], are determined implicitly by  $d\hat{T} = \kappa ds \hat{N}$  with  $\kappa \geq 0$ . Differentiating (3) yields

$$d\hat{T} = \frac{dz \hat{r} - dr \hat{z}}{a} = \frac{(-r/z \hat{r} - \hat{z}) dr}{a} = \frac{(-r/z \hat{r} - \hat{z}) ds}{a^2/z} = \frac{ds}{a} \left( -\frac{\vec{r}}{|\vec{r}|} \right), \quad (4)$$

so that  $\kappa = 1/a$ .

Our circle can be thought of as a line of constant longitude on the sphere, which produces the surface of revolution obtained by rotating the circle about the  $z$ -axis. What might we mean by the curvature of this surface?

We begin by rewriting the curvature  $\kappa$  as

$$\kappa ds = d\hat{T} \cdot \hat{N} = -\hat{T} \cdot d\hat{N} = +\hat{T} \cdot d\hat{n}$$

where the second equality follows from  $\hat{T} \cdot \hat{N} = 0$ , and where  $\hat{n} = -\hat{N}$  is the *outward* unit normal vector to the sphere. The resulting relation,  $\kappa ds = \hat{T} \cdot d\hat{n}$ , defines the *normal curvature*  $\kappa$  of a curve in a surface with normal vector  $\hat{n}$  [1, 5]. The normal curvature clearly depends on the choice of  $\hat{n}$ ;  $\kappa$  is no longer assumed to be nonnegative. By the *curvature* of a curve in a surface, we henceforth mean its normal curvature.

In this form, we can determine the curvature in the angular direction by replacing  $\hat{T}$  and  $ds$  with the unit tangent vector ( $\hat{\phi}$ ) and infinitesimal arclength ( $r d\phi$ ), respectively, along a line of latitude. Thus, this second curvature is

$$\kappa_2 = \hat{\phi} \cdot \frac{d\hat{n}}{ds} = \hat{\phi} \cdot \frac{d(r \hat{r} + z \hat{z})}{a(r d\phi)}.$$

The only  $\phi$ -dependence in  $\hat{n}$  is in  $\hat{r}$ ,<sup>1</sup> and a geometric argument [6] (or conversion to rectangular coordinates) shows directly that

$$d(r \hat{r}) = dr \hat{r} + r d\phi \hat{\phi},$$

from which it also follows by the product rule that  $d\hat{r} = d\phi \hat{\phi}$ . Thus,

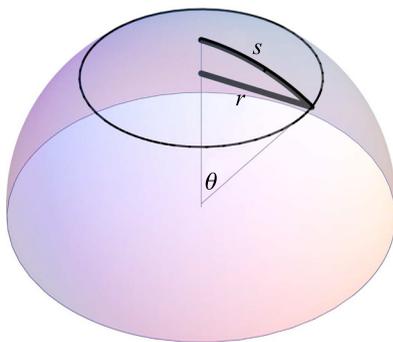
$$\kappa_2 = \hat{\phi} \cdot \frac{\hat{\phi}}{a} = \frac{1}{a}.$$

Although we have constructed our sphere explicitly as a surface of revolution in three-dimensional Euclidean space ( $\mathbb{R}^3$ ), the sphere itself ( $\mathbb{S}^2$ ) is a two-dimensional surface, with an intrinsic geometry that does not depend on how (or whether) it is embedded in Euclidean space. Running our construction backward, we could have started with the local geometry of  $\mathbb{S}^2$ , described by the line element

$$ds^2 = \frac{a^2 dr^2}{a^2 - r^2} + r^2 d\phi^2$$

(compare with (2)), then shown that this line element is the restriction of the Euclidean line element in  $\mathbb{R}^3$  (in cylindrical coordinates) if one sets  $r^2 + z^2 = a^2$ . Thus, (either diagram in) Figure 1 shows how to embed  $\mathbb{S}^2$  in  $\mathbb{R}^3$ . Such diagrams, typically with one

<sup>1</sup>On the line of longitude considered above,  $d\phi = 0$ , hence  $d\hat{r} = 0$ , as was assumed in (4).



**Figure 2.** The geometric radius  $r$  (heavy line) and the physical radius  $s$  (heavy arc) for a circle on (the upper half of) a sphere.

or more angular coordinates suppressed, are often used in relativity [17] to represent the shape of curved, three-dimensional surfaces, where they are called *embedding diagrams*.<sup>2</sup> In this context, it is important to realize that an embedding diagram introduces coordinates that are extrinsic to the surface one is analyzing. On the sphere,  $r$  and  $z$  are not independent; either one, but not both, can be used as a local coordinate (away from the equator).

As discussed further in Section 5, the curvatures  $\kappa_1 = \kappa$ ,  $\kappa_2$  are the *principal curvatures* of the sphere at the given point. Since the Gaussian curvature  $K$  is just the product of the two principal curvatures, the Gaussian curvature of the sphere is given by

$$K = \kappa_1 \kappa_2 = \frac{1}{a^2},$$

verifying the well-known result that the curvature of a sphere is a positive constant that is inversely proportional to the square of its radius. The *Theorema Egregium* of Gauss<sup>3</sup> shows that the Gaussian curvature, which we have constructed using an explicit embedding, is in fact an intrinsic property of the surface.

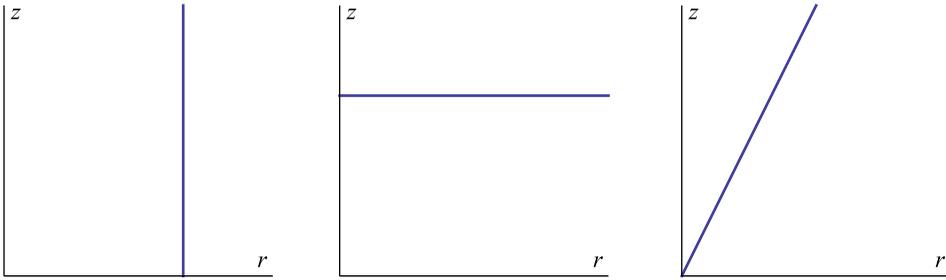
In this construction, although the principal curvature  $\kappa_1$  does indeed match the (original, multivariable-calculus notion of) curvature for a line of longitude, this equivalence does *not* hold for  $\kappa_2$  and a line of latitude. How did this happen? The normal vector  $\hat{n}$  is (minus) the principal normal vector  $\hat{N}$  for a line of longitude, but the principal normal vector for a line of latitude depends only on the curve, and is thus horizontal, therefore differing from  $\hat{n}$ , which depends on the surface.

What makes the sphere curved? Imagine looking down at a globe from above the North Pole. Lines of constant latitude will appear as concentric circles. What is the radius of such a circle? The obvious answer is, “the distance to the center.” But which distance? That is, which center?

The *geometric* center of a circle of latitude lies on the  $z$ -axis, and not on the sphere itself, as shown in Figure 2. The distance from this center to the circle is precisely our coordinate  $r$ , which we henceforth refer to as the *geometric radius*. Alternatively, *on the sphere*, the center of the circle lies at the North Pole; the *physical radius* needs to be computed using arclength  $s$  on the sphere, as also shown in Figure 2. Clearly, the geometric radius is less than the physical radius. Explicitly,  $s = a\theta$ , where  $\theta$  is

<sup>2</sup>The first use of this approach in relativity appears to be due to Fronsdal [10].

<sup>3</sup>The *Theorema Egregium* was originally published in 1827 in Latin, with English translations coming much later [11–13].



**Figure 3.** The generating curves for a cylinder, a plane, and a cone, all thought of as surfaces of revolution about the vertical ( $z$ ) axis.

colatitude (so  $r = a \sin \theta$ ). Eliminating  $\theta$  yields  $s = a \arcsin \frac{z}{a}$ , which could also have been obtained by integrating (2).

This, then, is one consequence of nonzero curvature: the geometric and physical radii are not the same. However, both notions of radius agree approximately for small circles, as can be seen geometrically from the limiting process as circles of latitude approach the North Pole.

**3. FLAT SURFACES.** Cylinders, planes, and cones are all generated as surfaces of revolution about the vertical axis by straight lines, as shown in Figure 3. In each case, the curvature  $\kappa_1$  is clearly 0. Explicitly, in each case we have

$$\hat{T} = \sin \alpha \hat{r} + \cos \alpha \hat{z} \tag{5}$$

where  $\alpha = \text{constant}$ , with  $\alpha = 0$  for the cylinder and  $\alpha = \frac{\pi}{2}$  for the plane. Assuming  $\alpha \in [0, \frac{\pi}{2}]$ , the outward normal vector is therefore

$$\hat{n} = \cos \alpha \hat{r} - \sin \alpha \hat{z} \tag{6}$$

so that

$$\kappa_1 ds = \hat{T} \cdot d\hat{n} = 0$$

since  $\hat{r}, \hat{z}$  do not depend on  $r$ . Along horizontal circles,  $ds = r d\phi$ , so we can also compute

$$\kappa_2 = \hat{\phi} \cdot \frac{d\hat{n}}{r d\phi} = \hat{\phi} \cdot \frac{\cos \alpha}{r} \hat{\phi} = \frac{\cos \alpha}{r}$$

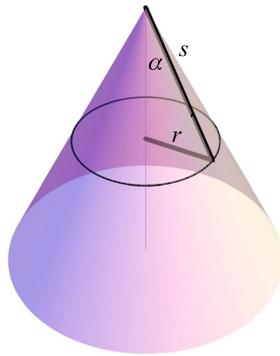
which is only 0 for the plane. However, the Gaussian curvature  $K = \kappa_1 \kappa_2$  vanishes in all cases.

What is the physical radius in each case? For the plane, the physical and geometric radii are clearly the same. For the cylinder, there is no physical radius, since no circle of latitude has a physical center! However, for the cone, the physical radius is the slant height, which we can determine starting from the line element (1), since

$$\frac{dz}{dr} = \cot \alpha$$

so that

$$ds^2 = (1 + \cot^2 \alpha) dr^2 = \frac{dr^2}{\sin^2 \alpha}.$$



**Figure 4.** The geometric radius  $r$  (horizontal line) and the physical radius  $s$  (slanted line) for a circle on a cone (shown upside-down from Figure 3).

Thus, the ratio of geometric radius to physical radius is the constant  $\sin \alpha$ , as shown in Figure 4. This failure of the two notions of radius to agree, even in the limit to the tip of the cone, signals the presence of a *conical singularity*.<sup>4</sup>

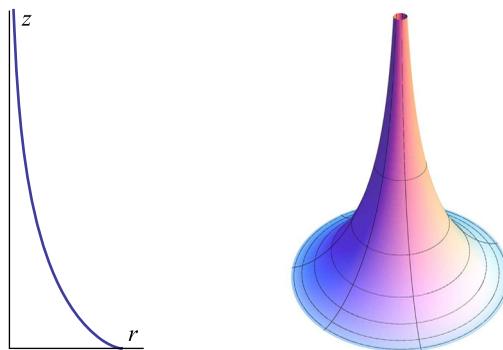
In all three cases, however, the physical and geometric radii are the same for all circles drawn around a physical center (except for circles on the cone that go around the tip).

**4. THE PSEUDOSPHERE.** The *tractrix*, shown in Figure 5, is the trajectory of an object being pulled by a rope that is initially along the horizontal axis, as the other end of the rope moves along the vertical axis. Such trajectories are called *pursuit curves*. Since the object always moves in the direction of the rope, the coordinates  $r, z$  must satisfy

$$\frac{dz}{dr} = -\frac{\sqrt{a^2 - r^2}}{r}$$

where  $a$  denotes the initial horizontal displacement. The solution of this differential equation is given by

$$z = -\sqrt{a^2 - r^2} - \frac{a}{2} \ln \frac{a - \sqrt{a^2 - r^2}}{a + \sqrt{a^2 - r^2}}$$



**Figure 5.** The tractrix (left), which generates the pseudosphere (right) as a surface of revolution.

<sup>4</sup>Conical singularities can be interpreted as distributional curvature.

which can also be described parametrically by

$$\begin{aligned} r &= a \sin \eta, \\ z &= -a \cos \eta - a \ln \tan \frac{\eta}{2}, \end{aligned}$$

where  $a = \text{constant}$  and  $\eta \in (0, \frac{\pi}{2}]$ . The *pseudosphere*, also shown in Figure 5, is the resulting surface of revolution.

Proceeding as for the sphere, we have

$$dz = -\frac{\sqrt{a^2 - r^2}}{r} dr$$

so that

$$ds^2 = dr^2 + dz^2 = \frac{a^2 dr^2}{r^2},$$

resulting in

$$\hat{\mathbf{T}} ds = d\vec{\mathbf{r}} = (\hat{\mathbf{r}} - \sqrt{a^2 - r^2} \hat{\mathbf{z}}/r) dr = \frac{(r \hat{\mathbf{r}} - \sqrt{a^2 - r^2} \hat{\mathbf{z}}) ds}{a}.$$

By inspection (compare (5) and (6)), the outward unit normal vector is therefore

$$\hat{\mathbf{n}} = \frac{\sqrt{a^2 - r^2} \hat{\mathbf{r}} + r \hat{\mathbf{z}}}{a},$$

which we can differentiate, yielding

$$d\hat{\mathbf{n}} = \left( -\frac{r \hat{\mathbf{r}}}{a\sqrt{a^2 - r^2}} + \frac{\hat{\mathbf{z}}}{a} \right) dr = -\frac{\hat{\mathbf{T}} dr}{\sqrt{a^2 - r^2}}.$$

Thus,

$$\kappa_1 = \hat{\mathbf{T}} \cdot \frac{d\hat{\mathbf{n}}}{ds} = -\frac{dr/ds}{\sqrt{a^2 - r^2}} = -\frac{r}{a\sqrt{a^2 - r^2}}.$$

Moving to the pseudosphere, as before,  $\hat{\mathbf{n}}$  depends on  $\phi$  only through  $\hat{\mathbf{r}}$ , so

$$\kappa_2 = \hat{\boldsymbol{\phi}} \cdot \frac{d\hat{\mathbf{n}}}{r d\phi} = \hat{\boldsymbol{\phi}} \cdot \frac{\sqrt{a^2 - r^2}}{ra} \hat{\boldsymbol{\phi}} = \frac{\sqrt{a^2 - r^2}}{ra}.$$

Combining these two curvatures, the Gaussian curvature of the pseudosphere is

$$K = \kappa_1 \kappa_2 = -\frac{1}{a^2};$$

the pseudosphere has constant *negative* curvature.

As with all surfaces of revolution, the geometric radius of a circle of constant latitude is  $r$ . But what is the physical radius of such circles? The physical center of each

such circle would be where  $r = 0$ , which only occurs at  $(0, \infty)$ ! Thus, the physical radius is

$$\int_{r=0}^r ds = \int_0^r \frac{a dr}{r} = \infty.$$

Although the “point” at infinity appears from Figure 5 to correspond to a conical singularity (with vertex angle 0), the ratio of circumference to radius is 0 everywhere, as is the ratio of geometric radius to physical radius.

Although requiring tools beyond those presented here, this ratio can also be determined for finite circles on the pseudosphere that do have a physical center. As with the sphere, the answer differs from 1; unlike the sphere, the ratio is greater than 1.

**5. SURFACES OF REVOLUTION.** We now generalize the above construction to any surface of revolution. The full line element is

$$ds^2 = dr^2 + r^2 d\phi^2 + dz^2, \quad (7)$$

with the surface given by specifying the relationship between  $r$  and  $z$ . Working first in the  $rz$ -plane, we have  $\phi = \text{constant}$ , so  $d\phi = 0$ . The unit tangent vector to a line of longitude is

$$\hat{T} ds = d\vec{r} = (\dot{r} \hat{r} + \dot{z} \hat{z}) ds,$$

where we have used dots to denote derivatives with respect to arclength, that is, with respect to  $s$ . Using negative reciprocal slopes, the outward unit normal vector must therefore be given by

$$\pm \hat{n} = \dot{z} \hat{r} - \dot{r} \hat{z},$$

where the sign depends on the sign of  $\dot{z}$ . Differentiating now leads to

$$\pm d\hat{n} = (\ddot{z} \hat{r} - \ddot{r} \hat{z}) ds$$

so that

$$\kappa_1 ds = \hat{T} \cdot d\hat{n} = \pm(\dot{r} \ddot{z} - \dot{z} \ddot{r}) ds.$$

Since  $\hat{n}$  only depends on  $\phi$  through  $\hat{r}$ , we also have

$$\kappa_2 = \hat{\phi} \cdot \frac{d\hat{n}}{r d\phi} = \pm \hat{\phi} \cdot \frac{\dot{z}}{r} \hat{\phi} = \pm \frac{\dot{z}}{r}.$$

However, since  $\hat{T}$  is a unit vector, we have

$$\dot{r}^2 + \dot{z}^2 = 1,$$

which implies that

$$\dot{r}\ddot{r} + \dot{z}\ddot{z} = 0.$$

Putting this all together, the Gaussian curvature of a surface of revolution is given by

$$K = \kappa_1 \kappa_2 = \frac{\dot{r}\ddot{z}\dot{z} - \dot{z}^2\ddot{r}}{r} = -\frac{(\dot{r}^2 + \dot{z}^2)\ddot{r}}{r} = -\frac{\ddot{r}}{r}. \quad (8)$$

Analogous formulas can be given using any other parameter along the curve, including  $r$  or  $z$ , as the independent variable instead of  $s$ . Such formulas are useful in practice, since the expression for arclength can be quite complicated. But it is often easiest to simply do the computation, as in the previous examples.

What features of the geometry are evident from the embedding diagrams? First, the most obvious: The sign of the Gaussian curvature  $K$  depends on the concavity of the surface. If the embedding diagram bends toward the axis ( $r$  concave down as a function of  $z$ ), as in [Figure 1](#), then  $K > 0$ ; if it bends away (concave up), as in [Figure 5](#), then  $K < 0$ ; if neither, as in [Figure 3](#), then  $K = 0$ .

Another obvious feature is that some of the generating curves intersect the  $z$ -axis, as in the cases of the circle and the cone, and others don't, as in the case of the cylinder. In the latter case, no notion of physical center exists.

The above computation can be generalized to any curve on the surface. Let  $u$  denote arclength along lines of longitude, so that

$$du^2 = dr^2 + dz^2.$$

The unit tangent vector  $\hat{U}$  along an arbitrary (smooth) curve in the surface must satisfy

$$\hat{U} = \hat{T} \cos \beta + \hat{\phi} \sin \beta,$$

where  $\beta$  is the angle between the curve and the longitudinal direction, so that in general

$$\tan \beta = \frac{r d\phi}{du}.$$

The curvature  $\kappa$  of the curve must satisfy

$$\kappa ds = \hat{U} \cdot d\hat{n},$$

and squaring both sides yields

$$\begin{aligned} \kappa^2 (du^2 + r^2 d\phi^2) &= \left( \hat{T} \cdot d\hat{n} \cos \beta + \hat{\phi} \cdot d\hat{n} \sin \beta \right)^2 \\ &= (\kappa_1 du \cos \beta + \kappa_2 r d\phi \sin \beta)^2 \end{aligned}$$

which simplifies to Euler's curvature formula, namely

$$\kappa = \kappa_1 \cos^2 \beta + \kappa_2 \sin^2 \beta. \quad (9)$$

One way to define the principal curvatures is as the extrema of  $\kappa$ , which are clearly  $\kappa_1$  and  $\kappa_2$ , as claimed in [Section 2](#); we have in fact shown that the principal directions for any surface of revolution lie along lines of latitude and longitude.

**6. THE SCHWARZSCHILD BLACK HOLE.** All of the previous examples were conceived as surfaces in Euclidean space. The original line element, either (1) or (7) in cylindrical coordinates, was restricted to the desired surface, yielding immediate expressions for arclength in each coordinate direction. We now consider an example that goes in the other direction.

The line element that now bears his name was originally discovered by Schwarzschild [19] as a spherically-symmetric, vacuum solution of Einstein's field equation; it turns out to be the unique solution with these properties. However, the interpretation of the coordinate  $r$  as the geometric radius is due to Droste [9], and the modern interpretation of this geometry as a black hole spacetime came 50 years after the initial discovery of the Schwarzschild line element.

The full Schwarzschild line element takes the form

$$ds^2 = -\left(1 - \frac{2m}{r}\right) dt^2 + \frac{dr^2}{1 - \frac{2m}{r}} + r^2 (d\theta^2 + \sin^2 \theta d\phi^2)$$

where we assume  $r > 2m$  and have followed standard practice in setting both the gravitational constant  $G$  and the speed of light  $c$  to 1. We consider here a  $t = \text{constant}$  slice of this geometry, representing an instant of "time," and further restrict to the equatorial plane ( $\theta = \frac{\pi}{2}$ ). The line element then reduces to the form

$$ds^2 = \frac{dr^2}{1 - \frac{2m}{r}} + r^2 d\phi^2. \tag{10}$$

In analogy with our previous examples, we seek to interpret this geometry as a surface in Euclidean space. Thus, we seek to express  $z$  in terms of  $r$  such that

$$dr^2 + dz^2 = \frac{dr^2}{1 - \frac{2m}{r}}. \tag{11}$$

It is straightforward to solve this differential equation. Since

$$\left(\frac{dz}{dr}\right)^2 = \frac{1}{1 - \frac{2m}{r}} - 1 = \frac{2m}{r - 2m},$$

we have

$$\frac{dz}{dr} = \frac{\sqrt{2m}}{\sqrt{r - 2m}},$$

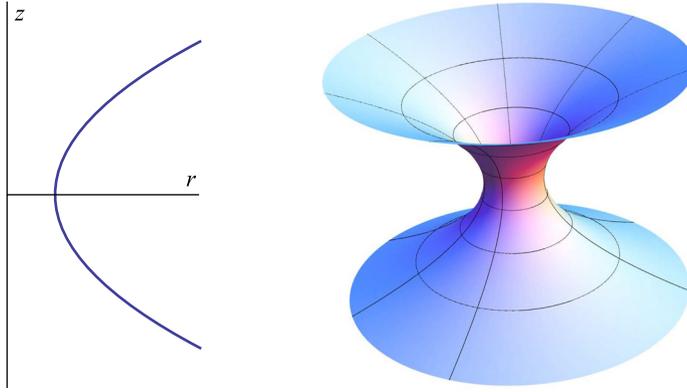
and therefore

$$z = \sqrt{2m} \int \frac{dr}{\sqrt{r - 2m}} = 2\sqrt{2m}\sqrt{r - 2m}$$

or equivalently

$$z^2 = 8mr - 16m^2,$$

which is a sideways parabola, as shown in Figure 6. The reader should be careful when interpreting this figure, as the coordinate "z" has no physical meaning; the geometry



**Figure 6.** The embedding diagram for a constant time slice of a Schwarzschild black hole, with one (right) or two (left) dimensions suppressed.

of our Schwarzschild slice is locally the same as (isometric to) the (rotated) parabola shown, but in this case the embedding is a mere convenience to help us understand the local geometry. And the reader should not forget that our slice is actually three-dimensional. By necessity, the right-hand diagram in Figure 6 suppresses an angular coordinate; each “circle” of constant latitude is really a sphere.

The Gaussian curvature of the resulting surface of revolution can be found directly as in the previous examples, or by using (8). Choosing the latter approach, we have from (11) that

$$\dot{r} = \sqrt{1 - \frac{2m}{r}},$$

so

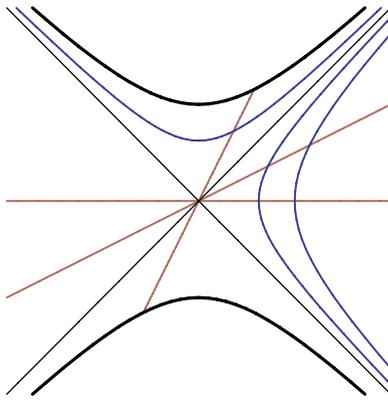
$$\ddot{r} = \frac{m/r^2}{\sqrt{1 - \frac{2m}{r}}} \dot{r},$$

leading finally to

$$K = -\frac{\ddot{r}}{r} = -\frac{m}{r^3}. \quad (12)$$

For the Schwarzschild geometry, the generating curve in Figure 6 clearly bends away; the Gaussian curvature is negative. Since the curve fails to meet the  $z$ -axis, circles of constant latitude have no physical center. But this geometry is supposed to model the gravitational field of a point mass! Where is it? Furthermore, as with the sphere, there are two points for each allowed value of  $r$ , with a single exception at the equator. For the sphere, we can understand this property as corresponding to two poles, each of which could serve as the physical center. For the Schwarzschild geometry, we instead have *two* asymptotic regions, where  $r$  can approach infinity (and where the geometry is approximately flat).

This surprising conclusion is in fact borne out by an analysis of the full, four-dimensional geometry, although this was not done until 1960 [16, 21]. Even though the line element (10) appears to be singular at the “throat” ( $r = 2m$ ), that singularity



**Figure 7.** The extended Kruskal geometry, with angular coordinates suppressed.

can be removed via a coordinate transformation, thus extending the underlying geometry; the resulting Kruskal geometry is shown in Figure 7. In this diagram, straight lines through the origin represent surfaces of constant  $t$ ; the hyperbolas represent surfaces of constant  $r$ , with the heavy lines at the top and bottom corresponding to  $r = 0$  and the asymptotes corresponding to  $r = 2m$ .

On any  $t = \text{constant}$  slice (with  $r > 2m$ ), there are indeed two asymptotic regions, connected at the “throat” where the radius is a minimum, exactly as implied by the embedding diagram in Figure 6. This feature is often called a *wormhole* [4, 15, 18], although it should be emphasized that it is not, in fact, possible to successfully traverse from one region to the other—there are no timelike trajectories (slope of modulus greater than one) connecting the left- and right-hand regions. The surface represented by the horizontal line, where  $t = 0$ , has intrinsic geometry corresponding to Figure 6; both diagrams show that  $r$  goes to  $\infty$  at both ends. However, Figure 7 is *not* an embedding diagram, as the vertical axis does not correspond to  $r = 0$ . Rather, each point in the diagram with  $r > 0$  represents an independent equatorial circle, corresponding to a two-sphere in four dimensions.

Finally, we note that it is possible to adapt our analysis to  $t = \text{constant}$  slices on which  $r < 2m$  using (10), although the geometry is no longer Riemannian. Nonetheless, the Gaussian curvature (12) correctly describes the behavior of the full, Riemann curvature near  $r = 0$ , namely that there is a curvature singularity there, with the curvature behaving like  $\frac{m}{r^3}$ .

In short, embedding diagrams capture the essence of the unexpectedly rich structure of this physically important geometry. These and other properties of a variety of spacetime geometries are further explored in most modern texts in general relativity, including [3, 6, 14, 17].

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#### REFERENCES

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- [1] Banchoff, T. F., Lovett, S. T. (2016). *Differential Geometry of Curves and Surfaces*, 2nd ed. Boca Raton, FL: A K Peters/CRC Press.

- [2] Briggs, W., Cochran, L., Gillett, B. (2015). *Calculus: Early Transcendentals*, 2nd ed. Boston, MA: Pearson.
- [3] Carroll, S. (2003). *Spacetime and Geometry: An Introduction to General Relativity*. Boston, MA: Pearson.
- [4] Collas, P., Klein, D. (2012). Embeddings and time evolution of the Schwarzschild wormhole. *Am. J. Phys.* 80(3): 203–210. [doi.org/10.1119/1.3672848](https://doi.org/10.1119/1.3672848)
- [5] do Carmo, M. P. (2016). *Differential Geometry of Curves and Surfaces*, revised ed. Mineola, NY: Dover Publications.
- [6] Dray, T. (2014). *Differential Forms and the Geometry of General Relativity*. Boca Raton, FL: A K Peters/CRC Press.
- [7] Dray, T., Manogue, C. A. (2003). Using differentials to bridge the vector calculus gap. *College Math. J.* 34(4): 283–290. [doi.org/10.2307/3595765](https://doi.org/10.2307/3595765)
- [8] Dray, T., Manogue, C. A. (2010). Putting differentials back into calculus. *College Math. J.* 41(2): 90–100. [doi.org/10.4169/074683410X480195](https://doi.org/10.4169/074683410X480195)
- [9] Droste, J. (1916). Het zwaartekrachtsveld van een of meer lichamen volgens de theorie van Einstein. Proefschrift (doctoral dissertation). Rijksuniversiteit te Leiden, The Netherlands.
- [10] Fronsdal, C. (1959). Completion and embedding of the Schwarzschild solution. *Phys. Rev.* 116(3): 778–781. [doi.org/10.1103/PhysRev.116.778](https://doi.org/10.1103/PhysRev.116.778)
- [11] Gauss, C. F. (1902). *General Investigations of Curved Surfaces of 1827 and 1825* (translated with notes and bibliography by James Caddall Morehead and Adam Miller Hildebeitel). Princeton, NJ: The Princeton University Library.
- [12] Gauss, C. F. (1965). *General Investigations of Curved Surfaces* (with an introduction by Richard Courant). Hewlett, NY: Raven Press.
- [13] Gauss, C. F. (2005). *General Investigations of Curved Surfaces* (edited with an introduction and notes by Peter Pesic). Mineola, NY: Dover Publications.
- [14] Hartle, J. B. (2003). *Gravity: An Introduction to Einstein's General Relativity*. San Francisco, CA: Addison–Wesley.
- [15] James, O., von Tunzelmann, E., Franklin, P., Thorne, K. S. (2015). Visualizing *Interstellar's* wormhole. *Am. J. Phys.* 83(6): 486–499. [doi.org/10.1119/1.4916949](https://doi.org/10.1119/1.4916949)
- [16] Kruskal, M. D. (1960). Maximal extension of Schwarzschild metric. *Phys. Rev.* 119(5): 1743–1745. [doi.org/10.1103/PhysRev.119.1743](https://doi.org/10.1103/PhysRev.119.1743)
- [17] Misner, C. W., Thorne, K. S., Wheeler, J. A. (1973). *Gravitation*. San Francisco: W. H. Freeman.
- [18] Müller, T. (2004). Visual appearance of a Morris–Thorne-wormhole. *Am. J. Phys.* 72(8): 1045–1050. [doi.org/10.1119/1.1758220](https://doi.org/10.1119/1.1758220)
- [19] Schwarzschild, K. (1916). Über das Gravitationsfeld eines Massenpunktes nach der Einsteinschen Theorie. *Sitzber. Deut. Akad. Wiss. Berlin.* 7: 189–196
- [20] Stewart, J. (2003). *Calculus: Early Transcendentals*, 5th ed. Belmont, CA: Brooks/Cole.
- [21] Szekeres, G. (1960). On the singularities of a Riemannian manifold. *Publ. Math. Debrecen.* 7: 285–301.

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