

Calculating Line Elements in Cylindrical and Spherical Coordinates

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Rectangular Coordinates:

The arbitrary infinitesimal displacement vector in Cartesian coordinates is:

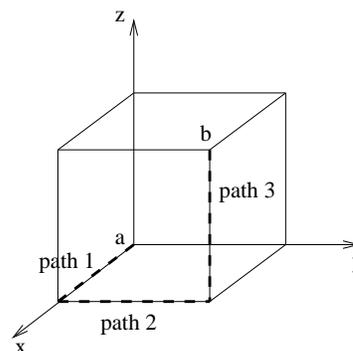
$$d\vec{r} = dx \hat{i} + dy \hat{j} + dz \hat{k}$$

Given the cube shown below, find $d\vec{r}$ on each of the three paths, leading from a to b .

Path 1: $d\vec{r} =$

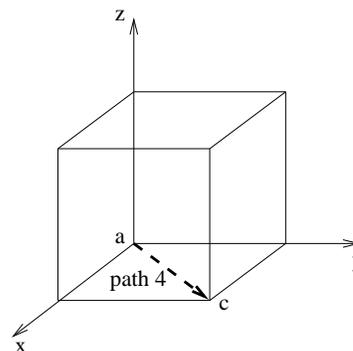
Path 2: $d\vec{r} =$

Path 3: $d\vec{r} =$



The first expression above for $d\vec{r}$ is valid for any path in rectangular coordinates. Find the appropriate expression for $d\vec{r}$ for the path which goes directly from a to c as drawn below.

Path 4: $d\vec{r} =$



However, Cartesian coordinates would be a **poor** choice to describe a path on a cylindrically or spherically shaped surface. Next we will find an appropriate expression in these cases.

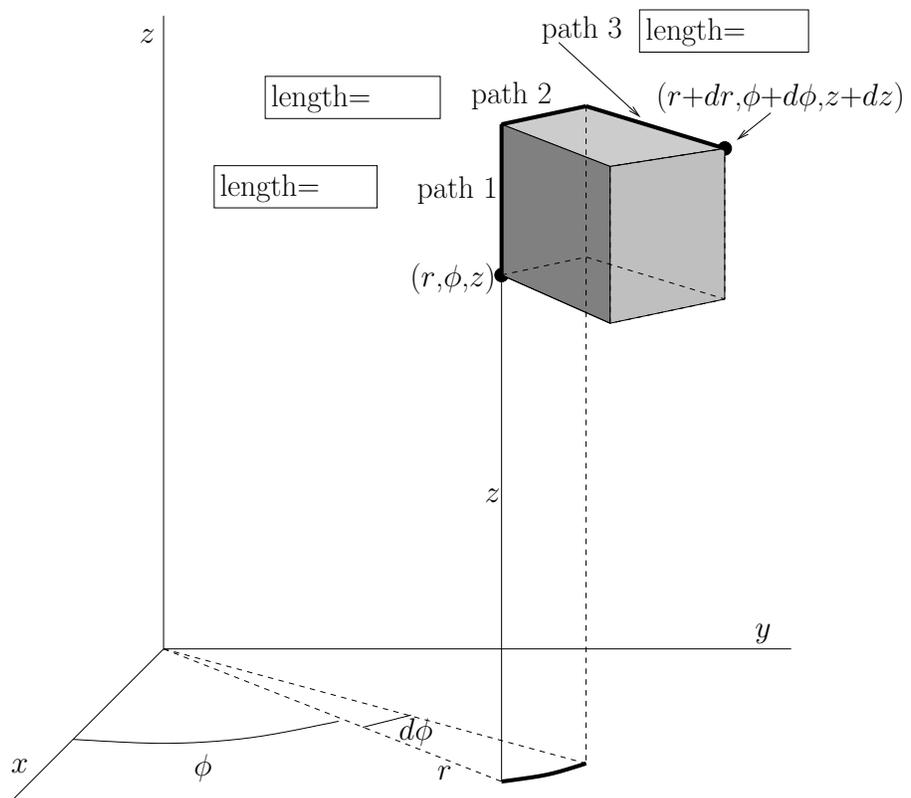
Cylindrical Coordinates:

You will now derive the general form for $d\vec{r}$ in cylindrical coordinates by determining $d\vec{r}$ along the specific paths below.

Note that an infinitesimal element of length in the \hat{r} direction is simply dr , just as an infinitesimal element of length in the \hat{i} direction is dx . **But**, an infinitesimal element of length in the $\hat{\phi}$ direction is **not** just $d\phi$, since this would be an angle and does not even have the units of length.

Geometrically determine the length of the three paths leading from a to b and write these lengths in the corresponding boxes on the diagram.

Now, remembering that $d\vec{r}$ has both magnitude and direction, write down below the infinitesimal displacement vector $d\vec{r}$ along the three paths from a to b . Notice that, along any of these three paths, only one coordinate r , ϕ , or z is changing at a time. (i.e. along path 1, $dz \neq 0$, but $d\phi = 0$ and $dr = 0$).



Path 1: $d\vec{r} =$

Path 2: $d\vec{r} =$

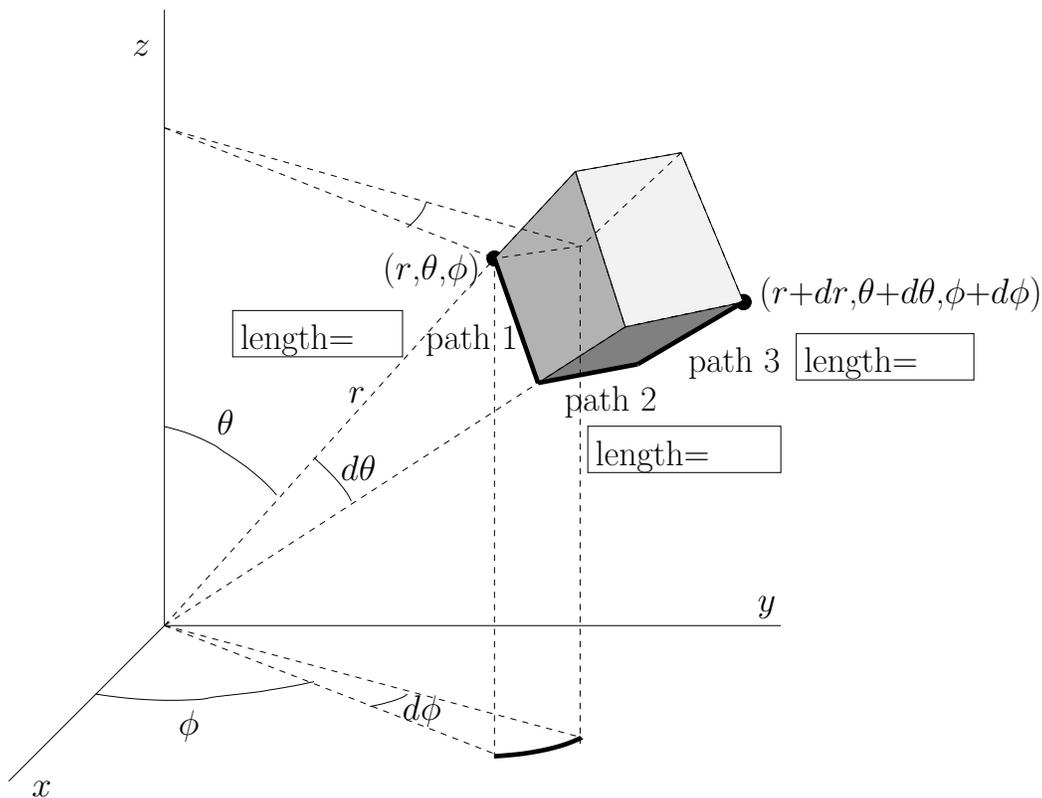
Path 3: $d\vec{r} =$

Cylindrical Coordinates

If all three coordinates are allowed to change simultaneously, by an infinitesimal amount, we could write this $d\vec{r}$ for any path as:

$$d\vec{r} =$$

This is the general line element in cylindrical coordinates.



Spherical Coordinates

Spherical Coordinates:

You will now derive the general form for $d\vec{r}$ in spherical coordinates by determining $d\vec{r}$ along the specific paths below. As in the cylindrical case, note that an infinitesimal element of length in the $\hat{\theta}$ or $\hat{\phi}$ direction is **not** just $d\theta$ or $d\phi$. You will need to be more careful. Geometrically determine the length of the three paths leading from a to b and write these lengths in the corresponding boxes on the diagram. Now, remembering that $d\vec{r}$ has both magnitude and direction, write down below the infinitesimal displacement vector $d\vec{r}$ along the three paths from a to b . Notice that, along any of these three paths, only one coordinate r , θ , or ϕ is changing at a time. (i.e. along path 1, $d\theta \neq 0$, but $dr = 0$ and $d\phi = 0$).

Path 1: $d\vec{r} =$

Path 2: $d\vec{r} =$

(Be careful, this is the tricky one.)

Path 3: $d\vec{r} =$

If all 3 coordinates are allowed to change simultaneously, by an infinitesimal amount, we could write this $d\vec{r}$ for any path as:

$$d\vec{r} =$$

This is the general line element in spherical coordinates.